On fractional plasma problems

Published version:

DOI:10.1088/1361-6544/aaba37

(Article begins on next page)

On fractional plasma problems

Gian Marco Canneori
Department of Mathematics
Università di Torino
Via Carlo Alberto 10, Torino, I-10123 Italy
gianmarco.canneori@unito.it,
Dimitri Mugnai*
Department of Ecology and Biology
University of Tuscia,
Via San Camillo de Lellis 44, Viterbo, I-01100 Italy
dimitri.mugnai@unitus.it

Abstract

In this paper we show existence and multiplicity results for a linearly perturbed elliptic problem driven by nonlocal operators, whose prototype is the fractional Laplacian. More precisely, when the perturbation parameter is close to one of the eigenvalues of the leading operator, the existence of three nontrivial solutions is proved.

Keywords: singular integral operator, eigenvalues, $(\nabla)-$condition, $\nabla-$theorem.
2000AMS Subject Classification: 35A15, 35B38, 35S15, 47G20, 45G05.

1 Introduction

A Tokamak machine consists of a toroidal cavity containing a given mass of plasma surrounded by a vacuum layer. One of the issues of the Plasma Physics deals with the specification of the region occupied by such a plasma at equilibrium and the description of its flux through a cross section $\Omega \subseteq \mathbb{R}^2$ of the machine, which we assume to be a bounded domain.

Denoting by u the flux function, a possible description of such a phenomenon is given by the nonlinear eigenvalue problem (though λ is not an eigenvalue

*Partially supported by the Italian MIUR project *Variational methods, with applications to problems in mathematical physics and geometry* (2015KB9WPT_009). D.M. is also member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and is supported by the 2017 INdAM-GNAMPA Project *Equazioni Differenziali Non Lineari*

according to the usual terminology)

$$(1) \qquad -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{1}{x_1}\frac{\partial u}{\partial x_i}\right) = \lambda u_+,$$

with specific boundary conditions and $\lambda \in \mathbb{R}$, see [12] and [28]. However, in [28] the domain Ω is far from the line $x_1 = 0$, and so the operator is uniformly elliptic. For this reason, a simplified but formally equivalent version of (1) is considered in [29], with the equation

$$-\Delta u = \lambda u_+.$$

However, in most cases, the Laplace operator does not fit the problem in a realistic way. Recall that, since the papers of Einstein [6] and Smoluchowski [25], the Laplacian has been the successful tool to describe diffusion and Brownian motion. However, the diffusion of a particle (or of an individual in a Biological species) at a point x might be influenced by *all* other particles, and this is particularly true if one takes into account also long–range particle interactions. For this reasons, in this case the diffusion operator cannot act pointwise, and so it is natural to consider the average between the total contributions, that is an integral average, of the form

$$\int_{\mathbb{R}^N} \big[u(x+y) - u(x)\big] K(y)\, dy,$$

where K is a weight which measures the influence of the particle at $x + y$ on the one at x. Typically, such influence is determined by singular interactions depending on the distance between the two points, and so a quantity of the form

$$(2) \qquad \int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^\alpha}\, dy$$

for some $\alpha > 0$ is the most natural operator which shows up, see [4], [26] and [30] for more details on this replacement of local operators by nonlocal ones.

To our best knowledge, the first replacement of a local operator with a nonlocal one was considered in [1], where the author uses the fractional Laplacian in the eigenvalue problem

$$(3) \qquad \begin{cases} A_s u = \lambda(u-a)_+ & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\Omega \subseteq \mathbb{R}^N$ a bounded domain, $u : \Omega \to \mathbb{R}$, $s \in (0,1)$ and $a \geq 0$ is a given constant. Here the fractional Laplacian taken into account is

$$A_s u \doteq \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{s-1}(\lambda - \Delta)^{-1} \Delta u \, d\lambda.$$

Note that, in spite of the nonlocal definition of A_s, it is now well known that a local representation is available: if $u \in H^2(\Omega) \cap H^1_0(\Omega)$, $u = \sum a_i\phi_i$, where

$a_i = \int_\Omega u\phi_i dx \in \mathbb{R}$ and ϕ_i are the L^2-orthonormalized eigenfunctions of $-\Delta$ in Ω with associated eigenvalues Λ_i, we have

$$A_s u = \sum_{i=1}^{\infty} \Lambda_i^s a_i \phi_i,$$

see [2]. For such a problem, Mark Allen studied existence and regularity of solutions and the properties of the free boundary, using the fact that this fractional Laplacian can be seen as a Dirichlet-to-Neumann map, which makes the original nonlocal problem in a local one with an additional dimension, see [5] for the entire space and [3] for bounded domains. For completeness, we briefly recall this Dirichlet-to-Neumann procedure: set $\mathcal{C} = \Omega \times (0, \infty)$, and, given u, consider the solution v of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v)(x,y) = 0 & (x,y) \in \mathcal{C}, \\ v(x,y) = 0 & x \in \partial\Omega, y > 0, \\ v(x,0) = u & x \in \Omega. \end{cases}$$

Then, there exists a constant $C_s > 0$ such that

$$C_s(-\Delta)^s u(x) = -\lim_{y\to 0^+} y^{1-2s} u_y(x,y), \quad x \in \Omega,$$

i.e the operator mapping the Dirichlet datum u to the Neumann–type datum $\lim_{y\to 0^+} -y^{1-2s}u_y(x,y)$ is $s-$root of the negative Laplacian $-\Delta$ in Ω. Also notice that when $s = 1/2$, we have $C_s = 1$ and $\sqrt{-\Delta}u(x) = -u_y(x,0)$. For the precise value of C_s, see [5].

However, it is clear from [2] that actually A_s has a *local* representation in terms of eigenvalues and eigenfunctions of $-\Delta$ in Ω, and thus *nonlocal interactions are not really considered. For this reason, in this paper* we start from different nonlocal versions of problem (3), whose prototype is

$$\begin{cases} (-\Delta)^s u = \lambda(u-a)_+ & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $a \geq 0$, $\lambda \in \mathbb{R}$, $u : \mathbb{R}^N \to \mathbb{R}$ and

$$(-\Delta)^s u(x) \doteq C(N,s)\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

in agreement with the Physical motivations which led to (2).

From now on Ω will be a bounded domain of $\mathbb{R}^N$ with Lipschitz continuous boundary. Note that the boundary condition "$u = 0$ on $\partial\Omega$" is replaced by the nonlocal one "$u = 0$ in $\mathbb{R}^N \setminus \Omega$, see [27] and [8, Theorem 4.4.3]. In this way, by definition of $(-\Delta)^s$, it is clear that an actual *nonlocal* operator is in force, and so nonlocal effects describing interactions among particles can be considered. It is also worth mentioning that A_s and $(-\Delta)^s$ are different operators with different eigenvectors and eigenfunctions, see [23]. However, there are also some very good properties that this operator enjoys, similarly to the Laplacian, or a uniformly elliptic operator: for instance, it admits a simple and positive principal eigenvalue with signed positive eigenfunction ([23]) and it satisfies the maximum principle and the Harnack inequality ([5]).

In this paper we shall consider more general problems of the form

$$(P) \qquad \begin{cases} \mathscr{L}_K u = \lambda (u-a)_+ & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$\mathscr{L}_K u(x) \doteq \int_{\mathbb{R}^N} \big(u(x) - u(y)\big) K(x-y)\, dy. \tag{4}$$

Here K is a singular potential, whose prototype is $K(x) = 1/|x|^{N+2s}$, see assumption $(\mathcal{H})$ for the precise setting, and u belongs to a suitable reference space X_0^s, see below.

Even without the precise definitions of the main characters involved, if we minimize the energy functional

$$\mathscr{E}(u) \doteq \iint_{\mathscr{O}} |u(x) - u(y)|^2 K(x-y)\, dxdy,$$

on the weakly closed constraint

$$\left\{ u \in X_0^s \,:\, \int_\Omega (u-a)_+^2\, dx = c > 0 \right\}$$

we find a minimizer u, with associated multiplier λ, so that the couple (u, λ) solves (P). In this way, we immediately find the counterpart of the existence result proved in [1] for the spectral fractional Laplacian:

Proposition 1.1. *If $a \geq 0$, then there exists $\lambda > 0$ such that problem (P) admits one solution (λ, u), $u \neq 0$.*

The fact that $\lambda > 0$ is proved simply starting from the identity

$$\iint_{\mathscr{O}} |u(x) - u(y)|^2 K(x-y)\, dxdy = \lambda \int_\Omega (u-a)_+ u\, dx,$$

using that $a \geq 0$.

In the rest of the paper we are interested in another version of problem (P), that is

$$(P_\lambda) \qquad \begin{cases} \mathscr{L}_K u = \lambda u - \gamma \left[(u+1)_-\right]^{p-1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $u : \mathbb{R}^N \to \mathbb{R}$, $\gamma > 0$ and $p > 2$. Notice that moving from (P) to (P_λ) we have set $a = 1$ (just to fix the ideas), and replaced u by $-u$. Of course, this choice is completely irrelevant and an analogous result can be proved for

$$\begin{cases} \mathscr{L}_K u = \lambda u + \gamma\,[(u-1)_+]^{p-1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

In order to introduce all the elements we need to solve problem (P_λ), we start recalling the usual setting for $\mathscr{L}_K$, see [22]. Take a function $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfying the assumption

($\mathcal{H}$). For $s \in (0,1)$ and $N > 2s$, we assume that

- $mK \in L^1(\mathbb{R}^{\mathbb{N}})$, with $m(x) \doteq \min\{|x|^2, 1\}$;
- $\exists\,\kappa > 0$ such that $K(x) \geq \kappa |x|^{-(N+2s)}$, for every $x \in \mathbb{R}^N \setminus \{0\}$;
- $K(-x) = K(x)$ for every $x \in \mathbb{R}^N \setminus \{0\}$.

Introduce the space

$$X^s \doteq \left\{ v : \mathbb{R}^N \to \mathbb{R} \;\middle|\; v_{|\Omega} \in L^2(\Omega),\ \big(v(x) - v(y)\big)\sqrt{K(x-y)} \in L^2(\mathscr{O}) \right\},$$

where $\mathscr{O} \doteq \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ and the space

$$X_0^s \doteq \left\{ u \in X^s \middle| u = 0 \text{ a.e. in } \mathbb{R}^{\mathbb{N}} \setminus \Omega \right\},$$

endowed with the scalar product

$$\langle u, v \rangle \doteq \iint_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) K(x-y)\, dx dy,$$

which makes X_0^s a Hilbert space, see [22]. From now on, we will denote by $\|\cdot\|$ the norm induced by $\langle \cdot, \cdot \rangle$.

Operator $\mathscr{L}_K$ is defined in (4) and $u \in X_0^s$ is a solution of (P_λ) if

$$\langle u, v \rangle = \lambda \int_\Omega uv\, dx - \gamma \int_\Omega [(u+1)_-]^{p-1}\, v\, dx$$

for every $v \in X_0^s$.

Setting

$$2^\star \doteq \frac{2N}{N - 2s},$$

our first easy result is

Theorem 1.2. *If $p \in (2, 2^\star)$, $\gamma > 0$, $\lambda \in \mathbb{R}$ and* **($\mathcal{H}$)** *holds, then (P_λ) admits one nontrivial solution.*

Finally, in order to give our main result, we recall that $\mathscr{L}_K$ admits a non-decreasing and diverging sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues with associated L^2-orthonormalized eigenfunctions $(e_n)_{n\in\mathbb{N}}$, such that, for every $u \in X_0^s$, we have

$$u = \sum_{n=1}^{\infty} \alpha_n e_n,$$

with $\alpha_n = \langle u, e_n\rangle \in \mathbb{R}$, for every $n \in \mathbb{N}$, see [24]. Moreover, $\lambda_1 > 0$ is simple, $e_1 > 0$ is bounded in Ω and every eigenvalue has finite multiplicity.

For further references, it is convenient to state the following remarks.

Remark 1.3. It is straightforward that $\{e_n\}_n$ are orthogonal also in X_0^s.

Remark 1.4. The negative (positive) part of any eigenfunction different from e_1 cannot be trivial on the whole domain. Indeed, if, by contraddiction, $(e_i)_- = 0$ in Ω for some $i \in \mathbb{N}$, $i \geq 2$, then

$$0 = \int_\Omega e_1 e_i \, dx = \int_\Omega e_1 (e_i)_+ \, dx > 0,$$

since $e_1 > 0$ in Ω.

Our main result is

Theorem 1.5. *Let $p \in (2, 2^\star)$, $\gamma > 0$ and* **($\mathcal{H}$)** *holds. If $l \in \mathbb{N}$ with $l \geq 1$, then there exists $\delta_l > 0$ such that, for every $\lambda \in (\lambda_{l+1} - \delta_l, \lambda_{l+1})$ problem (P_λ) admits three nontrivial solutions.*

The proof of this result is obtained by using a critical point theorem of mixed nature proved in [11], already successfully applied in [13], [14], [15], [16], [18], [19], [31], [32], also for variational inequalities, see [9].

2 Mathematical background

In this section we recall some results which will be used throughout the paper.

Lemma 2.1 (Lemma 9, [21])**.** *Assume that K satisfies* **($\mathcal{H}$)***. Then, the following assertions hold true:*

(i) *if Ω has a Lipschitz boundary, then the embedding $X_0^s \hookrightarrow L^p(\mathbb{R}^N)$ is compact, for every $p \in [1, 2^\star)$;*

(ii) *the embedding $X_0^s \hookrightarrow L^{2^\star}(\mathbb{R}^N)$ is continuous.*

Definition 2.2. Let X be a Banach space, $\mathscr{I} \in \mathscr{C}^1(X, \mathbb{R})$. We say that $\mathscr{I}$ satisfies the Palais-Smale condition, (PS) for short, if every $(u_n)_n \subseteq X$ such that $(\mathscr{I}(u_n))_n$ is bounded and $\mathscr{I}'(u_n) \to 0$ in X' admits a convergent subsequence. We say that $\mathscr{I}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, $(PS)_c$ for short, if every $(u_n)_n \subseteq X$ such that $\mathscr{I}(u_n) \to c$ and $\mathscr{I}'(u_n) \to 0$ in X' admits a convergent subsequence.

The following Linking Theorem, proved by Rabinowitz in [20], though well known, is here recalled in view of the estimate of the critical value, which will be needed for establishing the main result. As usual, S_ρ is the sphere of radius ρ in X and B_R is the ball of radius R.

Theorem 2.3. *Let X be a Banach space and $\mathscr{I} \in \mathscr{C}^1(X,\mathbb{R})$ be such that $\mathscr{I}(0) = 0$. Suppose that $X = X_1 \oplus X_2$, where X_1 and X_2 are closed subspaces with $\dim X_1 < \infty$. Assume that*

(i) *there exist $\rho, \alpha > 0$ such that*

$$\inf \mathscr{I}\left(S_\rho \cap X_2\right) = \alpha;$$

(ii) *there exists $e \in S_1 \cap X_2$ and $R > \rho$ such that, setting*

$$Q \doteq \left(\overline{B}_R \cap X_1\right) \oplus \{te \mid 0 < t < R\},$$

then $\mathscr{I}(\partial Q) \leq 0$;

(iii) *(PS) holds for $\mathscr{I}$.*

Then $\mathscr{I}$ has a critical value $\beta \geq \alpha$, where

$$\beta \doteq \inf_{h \in \mathscr{H}} \max_{v \in Q} \mathscr{I}\big(h(v)\big),$$

and

$$\mathscr{H} \doteq \{h \in \mathscr{C}(\overline{Q}, X) \,:\, h_{|\partial Q} = Id\}.$$

Finally, we state another critical point theorem, which is one of the ∇-*theorems* introduced by Marino and Saccon in [11]. The main feature of these theorems is the following condition, which essentially requires that the functional, constrained on a certain subspace, has no critical points with some uniformity.

Definition 2.4 (∇-condition)**.** Let X be a Hilbert space and $\mathscr{I} \in \mathscr{C}^1(X,\mathbb{R})$. Let C be a closed subspace of X and $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$.

We say that $\mathscr{I}$ verifies condition $(\nabla)(\mathscr{I}, C, a, b)$ if

$$\begin{aligned} &\text{there exists } \nu > 0 \text{ such that} \\ &\inf\Big\{\, \|P_C \nabla \mathscr{I}(u)\| \,:\, a \leq \mathscr{I}(u) \leq b,\ \operatorname{dist}(u, C) \leq \nu \Big\} > 0, \end{aligned}$$

where $P_C : X \to C$ denotes the orthogonal projection of X onto C.

Now we give the abstract theorem.

Theorem 2.5 (Theorem 2.10, [11])**.** *Let X be a Hilbert space and X_1, X_2, X_3 be three subspaces of X such that $X = X_1 \oplus X_2 \oplus X_3$, with $0 < \dim X_i < \infty$,* *for* $i = 1, 2$. *Let* $\mathscr{I} : X \to \mathbb{R}$ *be a* $\mathscr{C}^{1,1}$ *functional. Let* $\rho, \rho', \rho'', \rho_1$ *be such that* $0 < \rho_1$, $0 \le \rho' < \rho < \rho''$, *and set*

$$\Delta \doteq \{u \in X_1 \oplus X_2 : \rho' \le \|P_2 u\| \le \rho'', \|P_1 u\| \le \rho_1\}, \text{ and}$$
$$T \doteq \partial_{X_1 \oplus X_2} \Delta \text{ (the boundary of } \Delta \text{ in } X_1 \oplus X_2\text{)},$$

where $P_i : X \to X_i$ *denotes the orthogonal projection of* X *onto the subspace* X_i, *for every* $i = 1, 2$, *and*

$$S_{23}(\rho) \doteq \{u \in X_2 \oplus X_3 : \|u\| = \rho\},$$
$$B_{23}(\rho) \doteq \{u \in X_2 \oplus X_3 : \|u\| < \rho\}.$$

Assume that

$$a' = \sup \mathscr{I}(T) < \inf \mathscr{I}(S_{23}(\rho)) = a''.$$

Let a, b *be such that* $a' < a < a''$, $b > \sup \mathscr{I}(\Delta)$ *and*

$$(\nabla)(\mathscr{I}, X_1 \oplus X_3, a, b) \text{ holds;}$$
$$(PS)_c \text{ holds for } \mathscr{I}, \text{ for every } c \in [a, b].$$

Then, $\mathscr{I}$ *has at least two critical points in* $\mathscr{I}^{-1}([a, b])$.

3 The superlinear problem (P_λ)

In this section we shall prove Theorem 1.2. A first tool is the following

Lemma 3.1. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $p \ge 1$.
If $u \in L^p(\Omega)$, *then*

$$\int_\Omega [(u+1)_-]^p \, dx = o\left(\|u\|_p^p\right),$$

where $o\left(\|u\|_p^p\right) \to 0$ *as* $\|u\|_p \to 0$.

Proof. First of all,

$$\int_\Omega [(u+1)_-]^p \, dx = \int_{\{u \le -1\}} (-u-1)^p \, dx \le \int_{\{u \le -1\}} (-u)^p \, dx.$$

Now take a sequence $(u_n)_n \subseteq L^p(\Omega)$ such that $u_n \to 0$ in $L^p(\Omega)$ and any subsequence $(u_{n_k})_k$. Up to another subsequence, we can assume that $u_{n_k} \to 0$ a.e. in Ω. For every sub-subsequence $(u_{n_{k_j}})_j$ we get that

$$\begin{aligned}\int_\Omega \left[(u_{n_{k_j}}+1)_-\right]^p dx &\le \int_{\{u_{n_{k_j}} < -1\}} (-u_{n_{k_j}})^p \, dx \\ &\le \int_\Omega \left|u_{n_{k_j}}\right|^p \left|\{u_{n_{k_j}} < -1\}\right| dx = \|u_{n_{k_j}}\|_p^p \, o(1),\end{aligned}$$

since $u_{n_{k_j}} \to 0$ a.e. in Ω. This being valid for any sub-subsequence, we get the claim. ∎

Recalling the spectral properties of $\mathscr{L}_K$ described above, we have a standard decomposition of X_0^s. Indeed, if $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and

$$H_i \doteq \operatorname{span}\{e_1, \ldots, e_i\}$$

with its orthogonal complement

$$H_i^\perp \doteq \overline{\operatorname{span}\{e_{i+1} \ldots\}},$$

we obtain the decomposition

$$X_0^s = H_i \oplus H_i^\perp, \tag{5}$$

where $H_0 \doteq \{0\}$.

By using the previous notation, we get this useful result, for whose proof see [24].

Proposition 3.2. *Suppose that $u \in X_0^s$ and $i \in \mathbb{N}_0$.*

(i) *If $u \in H_i$, then*

$$\|u\|^2 \le \lambda_i \int_\Omega u^2\, dx; \tag{6}$$

(ii) *if $u \in H_i^\perp$, then*

$$\|u\|^2 \ge \lambda_{i+1} \int_\Omega u^2\, dx. \tag{7}$$

O course, problem (P_λ) has a variational structure, since it is the Euler-Lagrange equation of the $\mathscr{C}^1$ functional $\mathscr{F}_\lambda : X_0^s \to \mathbb{R}$, defined as

$$\mathscr{F}_\lambda(u) \doteq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega u^2\, dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\, dx. \tag{8}$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Throughout the proof we will adopt the decomposition

$$X_0^s = H_i \oplus H_i^\perp,$$

for some $i \in \mathbb{N}_0$, introduced at the beginning of the section.

First, we observe that when $\lambda = \lambda_1$, the family $\{te_1\}_{t \ge -\|e_1\|_\infty}$ defines a ray of solutions, as a simple calculation shows.

<u>Case $\lambda < \lambda_1$</u>. In this case it is enough to choose the representation $X_0^s = H_0^\perp$. Since X_0^s is continuously embedded in $L^2(\Omega)$ (see Lemma 2.1), we have that

$$\|u\|^2 - \lambda \int_\Omega u^2\, dx \le \|u\|^2 + |\lambda| \int_\Omega u^2\, dx \le c\|u\|^2,$$

with $c > 0$; moreover, by (7) we get that

$$\|u\|^2 - \lambda \int_\Omega u^2\,dx \geq \begin{cases} \left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 = C\|u\|^2 & \text{if } \lambda \in [0, \lambda_1), \\ \|u\|^2 & \text{if } \lambda < 0, \end{cases}$$

for some $C > 0$. Hence, in this case the norm defined as

$$\|u\|_\sim^2 \doteq \|u\|^2 - \lambda \int_\Omega u^2\,dx$$

is equivalent to the usual one $\|\cdot\|$. This said, let us check that $\mathscr{F}_\lambda$ satisfies the assumptions of the Mountain Pass Theorem.

In a sphere of radius $\rho > 0$ small enough, by Lemma 3.1 and Lemma 2.1, we get that

$$\begin{aligned} \mathscr{F}_\lambda(u) &= \frac{1}{2}\|u\|_\sim^2 - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\,dx \\ &\geq \frac{1}{2}\|u\|_\sim^2 - \frac{\gamma}{p}\varepsilon\|u\|_p^p \\ &\geq \frac{1}{2}\|u\|_\sim^2 - C_\varepsilon\|u\|_\sim^p = \|u\|_\sim^2\left(\frac{1}{2} - C_\varepsilon\|u\|_\sim^{p-2}\right) \geq \alpha > 0, \end{aligned}$$

provided that ε, $C_\varepsilon > 0$ and $\rho < (2C_\varepsilon)^{1/(2-p)}$, and so 0 is a strict local minimum point for $\mathscr{F}_\lambda$.

Now, by choosing $u < 0$ in Ω and $t > 0$ we have that

$$\begin{aligned} \mathscr{F}_\lambda(tu) &= \frac{t^2}{2}\|u\|_\sim^2 - \frac{\gamma}{p}\int_\Omega [(tu+1)_-]^p\,dx \\ &= \frac{t^2}{2}\|u\|_\sim^2 - \gamma\frac{t^p}{p}\int_\Omega \left[\left(u + \frac{1}{t}\right)_-\right]^p\,dx. \end{aligned}$$

Therefore, by the Generalized Lebesgue Theorem, we get that

$$\mathscr{F}_\lambda(tu) \underset{t\to\infty}{\longrightarrow} -\infty.$$

Finally, we need to prove the $(PS)_c$-condition. Take a sequence $(u_n)_n \subseteq X_0^s$ such that $\mathscr{F}_\lambda(u_n) \to c \in \mathbb{R}$ and such that $\mathscr{F}'_\lambda(u_n) \to 0$ in $(X_0^s)'$ as $n \to \infty$. Then there exist $M, N > 0$ such that

$$p\mathscr{F}_\lambda(u_n) - \mathscr{F}'_\lambda(u_n)u_n \leq M + N\|u_n\|_\sim.$$

On the other hand, we have

$$\begin{aligned} &p\mathscr{F}_\lambda(u_n) - \mathscr{F}'_\lambda(u_n)u_n \\ &\quad = \frac{p}{2}\|u_n\|_\sim^2 - \gamma\int_\Omega [(u_n+1)_-]^p\,dx - \|u_n\|_\sim^2 - \gamma\int_\Omega [(u_n+1)_-]^{p-1}\,u_n\,dx \\ &\quad = \left(\frac{p}{2} - 1\right)\|u_n\|_\sim^2 + \gamma\int_\Omega [(u_n+1)_-]^{p-1}\,dx \\ &\quad \geq \left(\frac{p}{2} - 1\right)\|u\|_\sim^2. \end{aligned}$$

Thus, since $p > 2$, it follows that $(u_n)_n$ is bounded in X_0^s.

Then, we get that, up to a subsequence, $u_n \rightharpoonup u$ in X_0^s and by Lemma 2.1

$$
\begin{aligned}
&u_n \to u \quad \text{in } L^p(\Omega) \text{ for every } p \in [2, 2^\star), \\
&u_n \to u \quad \text{a.e. in } \Omega.
\end{aligned} \tag{9}
$$

Since $(u_n - u)_n$ is a bounded sequence and $\mathscr{F}_\lambda'(u_n) \to 0$ in $(X_0^s)'$, we get that

$$\mathscr{F}_\lambda'(u_n)(u_n - u) \underset{n\to\infty}{\longrightarrow} 0;$$

but

$$\mathscr{F}_\lambda'(u_n)(u_n - u) = \|u_n\|_\sim^2 - \langle u_n, u\rangle_\sim + \gamma \int_\Omega [(u_n+1)_-]^{p-1}(u_n - u)\,dx$$

where

$$\langle u, v\rangle_\sim \doteq \langle u, v\rangle - \lambda \int_\Omega uv\,dx \text{ for every } u, v \in X_0^s.$$

By (9),

$$\int_\Omega [(u_n+1)_-]^{p-1}(u_n - u)\,dx \underset{n\to\infty}{\longrightarrow} 0,$$

thus we immediately get that $u_n \to u$ in X_0^s.

Hence, by the Mountain Pass Theorem, there exists a critical point $u \in X_0^s$ for $\mathscr{F}_\lambda$ with

$$\mathscr{F}_\lambda(u) \geq \alpha > 0,$$

that is problem (P_λ) admits one nontrivial solution, as well.

<u>Case $\lambda > \lambda_1$</u>. If $\lambda > \lambda_1$, then there exists $i \geq 1$ such that $\lambda_i \leq \lambda < \lambda_{i+1}$ and we shall apply Theorem 2.3 taking $X_1 = H_i$ and $X_2 = H_i^\perp$.

Suppose that $u \in H_i^\perp$. Then, in a sphere of radius ρ, by (7) and the same calculations used in the previous case, we have that

$$
\begin{aligned}
\mathscr{F}_\lambda(u) &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{i+1}}\right)\|u\|^2 - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\,dx \\
&\geq b\|u\|^2 - c\|u\|^p \\
&= \|u\|^2\left(b - c\|u\|^{p-2}\right) \geq \alpha > 0,
\end{aligned}
$$

with $b, c > 0$, provided that ρ is small enough.

Now, if $u \in H_i$, then, by (6), it follows that

$$\mathscr{F}_\lambda(u) \leq \frac{1}{2}(\lambda_i - \lambda)\int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\,dx \leq 0.$$

Moreover, taking $v = u + te_{i+1}$, with $u \in H_i$ and $t > 0$, since u and e_{i+1} are orthogonal in $L^2(\Omega)$ and in X_0^s (see Remark 1.3), by (6), we get that

$$\begin{aligned}\mathscr{F}_\lambda(v) &= \mathscr{F}_\lambda(u + te_{i+1})\\ &= \frac{1}{2}\|u + te_{i+1}\|^2 - \frac{\lambda}{2}\int_\Omega (u + te_{i+1})^2\,dx - \frac{\gamma}{p}\int_\Omega [(u + te_{i+1} + 1)_-]^p\,dx\\ &\le \frac{1}{2}(\lambda_i - \lambda)\int_\Omega u^2\,dx + \frac{t^2}{2}(\lambda_{i+1} - \lambda) - \frac{\gamma}{p}\int_\Omega [(u + te_{i+1} + 1)_-]^p\,dx\\ &\le \frac{t^2}{2}(\lambda_{i+1} - \lambda) - \gamma\frac{t^p}{p}\int_\Omega \left[\left(\frac{u}{t} + e_{i+1} + \frac{1}{t}\right)_-\right]^p dx.\end{aligned}$$

Using the Generalized Lebesgue Theorem, we get that

$$\int_\Omega \left[\left(\frac{u}{t} + e_{i+1} + \frac{1}{t}\right)_-\right]^p dx \underset{t\to\infty}{\longrightarrow} \int_\Omega (e_{i+1})_-^p\,dx,$$

and, since by Remark 1.4, $(e_{i+1})_- \not\equiv 0$, we get that $\mathscr{F}_\lambda(v) \to -\infty$ as $t \to \infty$ and (ii) of Theorem 2.3 holds, as well.

To conclude the proof, let us check that $\mathscr{F}_\lambda$ satisfies the $(PS)_c$-condition.

Given a $(PS)_c$-sequence $(u_n)_n \subseteq X_0^s$, there exists $M > 0$ such that $|\mathscr{F}_\lambda(u_n)| \le M$ and $\mathscr{F}'_\lambda(u_n) \to 0$ in $(X_0^s)'$.

Suppose by contradiction that $(u_n)_n$ is unbounded. Then, up to a subsequence, the sequence $\big(\|u_n\|\big)_n$ diverges, and by Lemma 2.1 we may assume that there exists $w \in X_0^s$ such that

$$\frac{u_n}{\|u_n\|} \rightharpoonup w \ \text{ in } X_0^s, \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{10}$$

Let us observe that (10) implies that

$$\frac{\mathscr{F}_\lambda(u_n)}{\|u_n\|^p} \underset{n\to\infty}{\longrightarrow} -\frac{\gamma}{p}\int_\Omega (w_-)^p\,dx$$

and thus, since $\mathscr{F}_\lambda(u_n)/\|u_n\|^p \to 0$ as $n \to \infty$, it follows that

$$w \ge 0 \text{ in } \Omega. \tag{11}$$

On the other hand, we can write

$$\frac{\mathscr{F}'_\lambda(u_n)u_n}{\|u_n\|} = 2\frac{\mathscr{F}_\lambda(u_n)}{\|u_n\|} + \gamma\left(\frac{2}{p} - 1\right)\int_\Omega \frac{[(u_n + 1)_-]^p}{\|u_n\|}\,dx - \gamma\int_\Omega \frac{[(u_n + 1)_-]^{p-1}}{\|u_n\|}\,dx,$$

where the first term of the right-hand-side goes to zero as $n \to \infty$ and the other two terms are non-positive. Hence, since $\mathscr{F}'_\lambda(u_n)u_n/\|u_n\| \to 0$ as $n \to \infty$, we have that both the following limits exist and

$$\lim_{n\to\infty}\int_\Omega \frac{[(u_n + 1)_-]^p}{\|u_n\|}\,dx = \lim_{n\to\infty}\int_\Omega \frac{[(u_n + 1)_-]^{p-1}}{\|u_n\|}\,dx = 0,$$

and consequently
$$\lim_{n\to\infty}\int_\Omega \frac{[(u_n+1)_-]^{p-1}\,u_n}{\|u_n\|}\,dx=0.$$
Therefore, by (10) we get that
$$\begin{aligned}\frac{\mathscr{F}_\lambda'(u_n)u_n}{\|u_n\|^2}&=1-\lambda\int_\Omega \frac{u_n^2}{\|u_n\|^2}\,dx+\gamma\int_\Omega \frac{[(u_n+1)_-]^{p-1}\,u_n}{\|u_n\|}\,dx\\&\underset{n\to\infty}{\longrightarrow} 1-\lambda\int_\Omega w^2\,dx.\end{aligned}$$
But $\mathscr{F}_\lambda'(u_n)u_n/\|u_n\|^2\to 0$ as $n\to\infty$, thus
$$w\not\equiv 0. \tag{12}$$

Moreover, if $v\in\mathscr{C}_c^\infty(\Omega)$, we obtain that
$$\frac{\mathscr{F}_\lambda'(u_n)v}{\|u_n\|}=\frac{\langle u_n,v\rangle}{\|u_n\|}-\lambda\int_\Omega \frac{u_n v}{\|u_n\|}\,dx+\gamma\int_\Omega \frac{[(u_n+1)_-]^{p-1}\,v}{\|u_n\|}\,dx,$$
but
$$\left|\int_\Omega \frac{[(u_n+1)_-]^{p-1}\,v}{\|u_n\|}\,dx\right|\le\|v\|_\infty\int_\Omega \frac{[(u_n+1)_-]^{p-1}}{\|u_n\|}\,dx\underset{n\to\infty}{\longrightarrow}0;$$
so, in this way, by (10)
$$\frac{\mathscr{F}_\lambda'(u_n)v}{\|u_n\|}\underset{n\to\infty}{\longrightarrow}\langle w,v\rangle-\lambda\int_\Omega wv\,dx,$$
for every $v\in\mathscr{C}_c^\infty(\Omega)$.

Now, since $\mathscr{F}_\lambda'(u_n)v/\|u_n\|\to 0$ as $n\to\infty$, it follows that
$$\langle w,v\rangle=\lambda\int_\Omega wv\,dx,\quad\text{for every } v\in\mathscr{C}_c^\infty(\Omega). \tag{13}$$

$\mathscr{C}_c^\infty(\Omega)$ being dense in X_0^s (see [7]), (13) holds for every v in X_0^s. Therefore, w is a nontrivial eigenfuction, see (12), of $\mathscr{L}_K$ with associated eigenvalue λ. If $\lambda\ne\lambda_i$, this is a contradiction.

On the other hand, if $\lambda=\lambda_i$, $w=\alpha e_i$, with $\alpha\in\mathbb{R}$, and a contradiction arises due to Remark 1.4 and the fact that $w\ge 0$, see (11).

Hence, every $(PS)_c$-sequence is bounded.

Therefore, by Lemma 2.1 we can suppose that
$$\begin{aligned}&u_n\rightharpoonup u\quad\text{in } X_0^s,\\&u_n\to u\quad\text{in } L^q(\Omega),\ \text{for any } q\in[1,2^\star),\\&u_n\to u\quad\text{a.e. in }\Omega.\end{aligned}\tag{14}$$

We can observe that the sequence $(u_n-u)_n$ is bounded and $\mathscr{F}_\lambda'(u_n)\to 0$ in $(X_0^s)'$, hence
$$\mathscr{F}_\lambda'(u_n)(u_n-u)\underset{n\to\infty}{\longrightarrow}0.$$

Since

$$\mathscr{F}'_\lambda(u_n)(u_n-u) = \|u_n\|^2 - \langle u_n, u\rangle - \lambda\int_\Omega u_n(u_n-u)\,dx + \gamma\int_\Omega [(u_n+1)_-]^{p-1}\,(u_n-u)\,dx;$$

by (14) it follows that $u_n \to u$ in X_0^s.

Therefore, by Theorem 2.3, there exists a nontrivial critical point u for $\mathscr{F}_\lambda$ with $\mathscr{F}_\lambda(u) > 0$, and thus problem (P_λ) admits a nontrivial solution. □

4 Multiplicity via ∇-theorems

The aim of this section is to produce a multiplicity result for the problem (P_λ). The idea is to apply Theorem 2.5 to the functional associated to (P_λ), which reads

$$\mathscr{F}_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\,dx,$$

for every $u \in X_0^s$.

Remark 4.1. Given $q \in [1, 2^\star)$, we define the inverse of the operator $\mathscr{L}_K$, $\mathscr{L}_K^{-1} : L^{q'}(\Omega) \to X_0^s$, as $\mathscr{L}_K^{-1} g = v$ if and only if $v \in X_0^s$ solves the problem

$$\begin{cases} \mathscr{L}_K v = g & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

so that

$$\langle u, \mathscr{L}_K^{-1} v\rangle = \int_\Omega uv\,dx, \quad \text{for every } u, v \in X_0^s. \tag{15}$$

Moreover, $\mathscr{L}_K^{-1} : L^{q'}(\Omega) \to X_0^s$ is compact.

Remark 4.2. Consider the Gâteaux derivative of $\mathscr{F}_\lambda$

$$\mathscr{F}'_\lambda(u)v = \langle u, v\rangle - \lambda\int_\Omega uv\,dx + \gamma\int_\Omega [(u+1)_-]^{p-1}\,v\,dx,$$

with $u, v \in X_0^s$. Then

$$\langle \nabla\mathscr{F}_\lambda(u), v\rangle = \mathscr{F}'_\lambda(u)v = \langle u, v\rangle - \langle \mathscr{L}_K^{-1}(\lambda u - \gamma\,[(u+1)_-]^{p-1}), v\rangle,$$

see [14].

If $l, m \in \mathbb{N}$, with $l \geq 1$ and $m \geq l+1$, are such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$, we will choose

$$\begin{aligned} X_1 &= H_l = \operatorname{span}\{e_1, \ldots, e_l\} \\ X_2 &= \operatorname{span}\{e_{l+1}, \ldots, e_m\} \\ X_3 &= H_m^\perp = \overline{\operatorname{span}\{e_{m+1}, e_{m+2}\ldots\}}, \end{aligned}$$

as functional setting for Theorem 2.5.

Definition 4.3. For any $i \geq 2$ define

$$\lambda_i^* \doteq \sup \left\{ \|u\|^2 : \ u \in H_i, \ u \geq 0 \text{ in } \Omega, \ \int_\Omega u^2 \, dx = 1 \right\}.$$

We immediately observe that the previous definition is well-posed, since $e_1 \in H_i$ for every $i \in \mathbb{N}$ and $e_1 > 0$ in Ω.

Remark 4.4. It is clear by Definition 4.3 that

$$\|u\|^2 \leq \lambda_i^* \int_\Omega u^2 \, dx,$$

for every $u \in X_i$, $u \geq 0$.

In the sequel, we will need the next property.

Proposition 4.5. *If $i \geq 2$, then $\lambda_i^* < \lambda_i$.*

Proof. Fix $i \geq 2$ and take $u \in H_i$ such that $u \geq 0$ in Ω and $\int_\Omega u^2 \, dx = 1$. If $u = \sum_{j=1}^{i} \alpha_j e_j$, with $\alpha_j = \int_\Omega u e_j \, dx$ for all $j = 1, \dots, i$ we have

$$1 = \int_\Omega u^2 \, dx = \sum_{j=1}^{i} \alpha_j^2.$$

Therefore, since $e_1 > 0$ in Ω, we get that $\alpha_1 = \int_\Omega u e_1 \, dx > 0$ and thus

$$\|u\|^2 = \sum_{j=1}^{i} \lambda_j \alpha_j^2 = \lambda_1 \alpha_1^2 + \sum_{j=2}^{i} \lambda_j \alpha_j^2 \leq \lambda_1 \alpha_1^2 + \lambda_i (1 - \alpha_1^2) < \lambda_i.$$

□

4.1 Geometry of the ∇-theorem

In this section we check that the geometric condition required in Theorem 2.5 holds true.

Proposition 4.6 ((∇)-geometry for $\mathscr{F}_\lambda$)**.** *Let $l, m \in \mathbb{N}$, with $l \geq 1$ and $m \geq l + 1$, be such that $\lambda_l < \lambda < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$, with $\lambda > \lambda_m^*$. Then, there exist ρ and R with $R > \rho > 0$, such that*

$$\sup_{\{u \in X_1 : \|u\| \leq R\} \cup \{u \in X_1 \oplus X_2 : \|u\| = R\}} \mathscr{F}_\lambda(u) < \inf_{\{u \in X_2 \oplus X_3 : \|u\| = \rho\}} \mathscr{F}_\lambda(u).$$

Proof. First of all, let us observe that $X_1 \oplus X_2 = H_m$ and $X_2 \oplus X_3 = H_l^\perp$. Thus, the idea is to show that there exist $R > \rho > 0$ such that

(a) $\inf_{\{u \in H_l^\perp : \|u\| = \rho\}} \mathscr{F}_\lambda(u) > 0$;

(b) $$\sup_{\{u\in H_l\colon \|u\|\le R\}\cup\{u\in H_m\colon \|u\|=R\}} \mathscr{F}_\lambda(u)=0.$$

Concerning (a), consider $u\in H_l^\perp$. Fixed $\varepsilon>0$, there exists $\rho>0$ small enough such that, by Lemma 3.1, (7) and Lemma 2.1, we have that

$$\begin{aligned}\mathscr{F}_\lambda(u)&=\frac{1}{2}\|u\|^2-\frac{\lambda}{2}\int_\Omega u^2\,dx-\frac{\gamma}{p}\int_\Omega[(u+1)_-]^p\,dx\\&\ge\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{l+1}}\right)\|u\|^2-\frac{\gamma}{p}\varepsilon\|u\|_p^p\\&\ge\|u\|^2\left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{l+1}}\right)-C_\varepsilon\|u\|^{p-2}\right)>0,\end{aligned}$$

for $\|u\|=\rho$ small enough.

Now we prove (b). First, consider $u\in H_l$. As a consequence of (6), we easily get that

$$\mathscr{F}_\lambda(u)\le\frac{1}{2}(\lambda_l-\lambda)\int_\Omega u^2\,dx\le 0.$$

On the other hand, if $u\in H_m$ it can be written in the form

$$u=v+w,$$

with $v\in H_l$ and $w\in\operatorname{span}\{e_{l+1},\dots,e_m\}$.

Suppose by contradiction that there exists a sequence $(v_j+w_j)_j$, with $v_j\in H_l$ and $w_j\in\operatorname{span}\{e_{l+1},\dots,e_m\}$ for every $j\in\mathbb{N}$, such that $\|v_j+w_j\|\to\infty$ as $j\to\infty$ and

$$\begin{aligned}&\mathscr{F}_\lambda(v_j+w_j)\\&\quad=\frac{1}{2}\|v_j+w_j\|^2-\frac{\lambda}{2}\int_\Omega(v_j+w_j)^2\,dx-\frac{\gamma}{p}\int_\Omega[(v_j+w_j+1)_-]^p\,dx>0.\end{aligned}$$

Since $H_m=H_l\oplus\operatorname{span}\{e_{l+1},\dots,e_m\}$ is finite-dimensional, then, up to a subsequence, by Lemma 2.1 we have that

$$\frac{v_j+w_j}{\|v_j+w_j\|}\to\bar v+\bar w\quad\text{in } X_0^s,\ L^q(\Omega)\text{ and a.e. in }\Omega, \tag{16}$$

for every $q\in[1,2^\star)$, with $\bar v\in H_l$ and $\bar w\in\operatorname{span}\{e_{l+1},\dots,e_m\}$. In particular, we have that

$$\bar v+\bar w\in H_m\text{ and }\|\bar v+\bar w\|=1. \tag{17}$$

If we divide $\mathscr{F}_\lambda(v_j+w_j)$ by $\|v_j+w_j\|^p$ we get that

$$\begin{aligned}0&<\frac{1}{2}\|v_j+w_j\|^{2-p}-\frac{\lambda}{2}\|v_j+w_j\|^{2-p}\int_\Omega\left(\frac{v_j+w_j}{\|v_j+w_j\|}\right)^2dx\\&\quad-\frac{\gamma}{p}\int_\Omega\left[\left(\frac{v_j+w_j}{\|v_j+w_j\|}+\frac{1}{\|v_j+w_j\|}\right)_-\right]^p dx\\&=o(1)-\frac{\gamma}{p}\int_\Omega[(\bar v+\bar w)_-]^p\,dx,\end{aligned}$$

with $o(1) \to 0$ as $j \to \infty$, and thus

$$\bar{v} + \bar{w} \geq 0 \text{ in } \Omega. \tag{18}$$

Moreover, from (18), (17) and Remark 4.4 we get

$$0 \leq \mathscr{F}_\lambda(\bar{v} + \bar{w}) = \frac{1}{2} - \frac{\lambda}{2} \int_\Omega (\bar{v} + \bar{w})^2 \, dx \leq \frac{1}{2}(\lambda_m^* - \lambda) \int_\Omega (\bar{v} + \bar{w})^2 \, dx < 0,$$

which is impossible. □

4.2 The (∇)-condition

This section is devoted to the proof of a suitable (∇)-condition for functional $\mathscr{F}_\lambda$. First of all, we need to prove two lemmas.

Lemma 4.7. *Let $l, m \in \mathbb{N}$, with $l \geq 1$ and $m \geq l+1$, be such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$. Then, for every $\sigma > 0$ there exists $\varepsilon_\sigma > 0$ such that, for every $\lambda \in [\lambda_l + \sigma, \lambda_{m+1} - \sigma]$, the unique critical point u of $\mathscr{F}_\lambda$ constrained on $H_l \oplus H_m^\perp$ such that $\mathscr{F}_\lambda(u) \in [-\varepsilon_\sigma, \varepsilon_\sigma]$, is the trivial one.*

Proof. We proceed by contradiction, so we suppose that there exist $\bar{\sigma} > 0$, $(\mu_j)_j \subseteq \mathbb{R}$, with $\mu_j \in [\lambda_l + \bar{\sigma}, \lambda_{m+1} - \bar{\sigma}]$, and a sequence $(u_j)_j \subseteq H_l \oplus H_m^\perp \setminus \{0\}$ such that

$$\mathscr{F}'_{\mu_j}(u_j)\varphi = 0, \text{ for every } \varphi \in H_l \oplus H_m^\perp \text{ and for every } j \in \mathbb{N}, \tag{19}$$

and

$$\mathscr{F}_{\mu_j}(u_j) = \frac{1}{2}\|u_j\|^2 - \frac{\mu_j}{2} \int_\Omega u_j^2 \, dx - \frac{\gamma}{p} \int_\Omega [(u_j + 1)_-]^p \, dx \underset{j \to \infty}{\longrightarrow} 0. \tag{20}$$

Since $u_j \in H_l \oplus H_m^\perp$ for every $j \in \mathbb{N}$, by (19) we have that

$$\begin{aligned}
0 &= \mathscr{F}'_{\mu_j}(u_j)u_j \\
&= \|u_j\|^2 - \mu_j \int_\Omega u_j^2 \, dx + \gamma \int_\Omega [(u_j + 1)_-]^{p-1} (u_j + 1 - 1) \, dx \\
&= 2\mathscr{F}_{\mu_j}(u_j) + \gamma \frac{2}{p} \int_\Omega [(u_j + 1)_-]^p \, dx - \gamma \int_\Omega [(u_j + 1)_-]^p \, dx \\
&\quad - \gamma \int_\Omega [(u_j + 1)_-]^{p-1} \, dx \\
&\leq 2\mathscr{F}_{\mu_j}(u_j) + \gamma \left(\frac{2}{p} - 1\right) \int_\Omega [(u_j + 1)_-]^p \, dx.
\end{aligned}$$

Being this valid for every $j \in \mathbb{N}$, as a consequence of (20) we get that

$$0 \leq \gamma \left(1 - \frac{2}{p}\right) \int_\Omega [(u_j + 1)_-]^p \, dx \leq 2\mathscr{F}_{\mu_j}(u_j) \underset{j \to \infty}{\longrightarrow} 0,$$

and so

$$\lim_{j\to\infty}\int_\Omega [(u_j+1)_-]^p\, dx = 0. \tag{21}$$

Now, since $u_j \in H_l \oplus H_m^\perp$ for every $j \in \mathbb{N}$, we can write

$$u_j = v_j + w_j,$$

with $v_j \in H_l$ and $w_j \in H_m^\perp$. Therefore, v_j and w_j are orthogonal in X_0^s, and so

$$\langle u_j, v_j\rangle = \|v_j\|^2,\ \langle u_j, w_j\rangle = \|w_j\|^2 \tag{22}$$

and

$$\|u_j\|^2 = \|v_j + w_j\|^2 = \|v_j\|^2 + \|w_j\|^2 = \|v_j - w_j\|^2. \tag{23}$$

Taking $\varphi = v_j - w_j \in H_l \oplus H_m^\perp$, by (19) we can write

$$\begin{aligned}0 &= \mathscr{F}'_{\mu_j}(u_j)(v_j - w_j)\\ &= \|v_j\|^2 - \|w_j\|^2 - \mu_j\int_\Omega v_j^2\,dx + \mu_j\int_\Omega w_j^2\,dx\\ &\quad + \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx,\end{aligned}$$

and so, by (7), (6), (22) and (23), we get

$$\begin{aligned}0 &= \|v_j\|^2 - \|w_j\|^2 - \mu_j\int_\Omega v_j^2\,dx + \mu_j\int_\Omega w_j^2\,dx\\ &\quad + \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx\\ &\le \|v_j\|^2 - \|w_j\|^2 - \frac{\mu_j}{\lambda_l}\|v_j\|^2 + \frac{\mu_j}{\lambda_{m+1}}\|w_j\|^2\\ &\quad + \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx\\ &= \frac{\lambda_l - \mu_j}{\lambda_l}\|v_j\|^2 + \frac{\mu_j - \lambda_{m+1}}{\lambda_{m+1}}\|w_j\|^2 + \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx\\ &\le -\frac{\bar\sigma}{\lambda_l}\|v_j\|^2 - \frac{\bar\sigma}{\lambda_{m+1}}\|w_j\|^2 + \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx\\ &\le -\frac{\bar\sigma}{\lambda_{m+1}}\|u_j\|^2 + \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx.\end{aligned}$$

In this way, we have

$$\frac{\bar\sigma}{\lambda_{m+1}}\|u_j\|^2 \le \gamma\int_\Omega [(u_j+1)_-]^{p-1}(v_j - w_j)\,dx. \tag{24}$$

On other hand, by Hölder's inequality, Theorem 2.1 and (23), we have that

$$\begin{aligned}\left|\int_\Omega [(u_j+1)_-]^{p-1}(v_j-w_j)\,dx\right| &\le \int_\Omega [(u_j+1)_-]^{p-1}|v_j-w_j|\,dx\\ &\le \left(\int_\Omega [(u_j+1)_-]^p\,dx\right)^{\frac{p-1}{p}}\|v_j-w_j\|_p\\ &\le C\left(\int_\Omega [(u_j+1)_-]^p\,dx\right)^{\frac{p-1}{p}}\|v_j-w_j\|\\ &\le C\left(\int_\Omega [(u_j+1)_-]^p\,dx\right)^{\frac{p-1}{p}}\|u_j\|.\end{aligned}$$

Thus, combining the previous inequality with (24) and the fact that $u_j \not\equiv 0$, we find that

$$\|u_j\| \le \tilde{C}\left(\int_\Omega [(u_j+1)_-]^p\,dx\right)^{\frac{p-1}{p}},\ \forall\, j\in\mathbb{N}, \tag{25}$$

with $\tilde{C} = \dfrac{\gamma C\lambda_{m+1}}{\bar{\sigma}}$.

Therefore, $\|u_j\|\to 0$ as $j\to\infty$, and thus, since $H_l\oplus H_m$ is finite-dimensional,

$$u_j \to 0 \text{ in } X_0^s. \tag{26}$$

But, by (25), Lemma 3.1 and Lemma 2.1, for every $\varepsilon>0$ we have that

$$\|u_j\| \le \tilde{C}\left(\int_\Omega [(u_j+1)_-]^p\right)^{\frac{p-1}{p}} \le \varepsilon\tilde{C}\|u_j\|^{p-1},$$

and thus $\|u_j\|^{p-2}>0$ for every $j\in\mathbb{N}$, which is in contradiction with (26). $\square$

Lemma 4.8. *Let $l,m\in\mathbb{N}$, with $l\ge 1$ and $m\ge l+1$, be such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$ and let $\lambda\in\mathbb{R}$, $\lambda\neq\lambda_1$. Denote by $P: X_0^s \to \operatorname{span}\{e_{l+1},\ldots,e_m\}$ and $Q: X_0^s\to H_l\oplus H_m^\perp$ the orthogonal projections.*

If $(u_j)_j$ is a sequence in X_0^s such that

- (i) *$(\mathscr{F}_\lambda(u_j))_j$ is bounded in $\mathbb{R}$;*
- (ii) *$Pu_j \underset{j\to\infty}{\longrightarrow} 0$ in X_0^s;*
- (iii) *$Q\nabla\mathscr{F}_\lambda(u_j) \underset{j\to\infty}{\longrightarrow} 0$ in X_0^s,*

then $(u_j)_j$ is bounded in X_0^s.

Proof. Suppose by contradiction that $(u_j)_j$ is unbounded in X_0^s. Then, up to a subsequence, $\|u_j\|\to\infty$ as $j\to\infty$ and so we may assume that there exists $u\in X_0^s$ such that (see Lemma 2.1)

$$\frac{u_j}{\|u_j\|} \rightharpoonup u \quad \text{in } X_0^s, \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{27}$$

By (i) and (27), dividing $\mathscr{F}_\lambda(u_j)$ by $\|u_j\|^p$, we get that

$$\begin{aligned} 0 \underset{j\to\infty}{\longleftarrow} \frac{\mathscr{F}_\lambda(u_j)}{\|u_j\|^p} &= \frac{1}{2}\|u_j\|^{2-p} - \frac{\lambda}{2}\|u_j\|^{2-p}\int_\Omega \left(\frac{u_j}{\|u_j\|}\right)^2 dx \\ &\quad - \frac{\gamma}{p}\int_\Omega \left[\left(\frac{u_j}{\|u_j\|} + \frac{1}{\|u_j\|}\right)_-\right]^p dx \\ &= o(1) - \frac{\gamma}{p}\int_\Omega (u_-)^p\, dx, \end{aligned}$$

with $o(1) \to 0$ as $j \to \infty$, and thus

$$u \geq 0 \text{ in } \Omega. \tag{28}$$

Moreover, every u_j can be written in the form

$$u_j = Pu_j + Qu_j. \tag{29}$$

Observe that

$$\langle Pu, v\rangle = \langle u, Pv\rangle, \quad \text{for every } u, v \in X_0^s, \tag{30}$$

and that

$$\int_\Omega Pu_j u_j\, dx = \int_\Omega (Pu_j)^2\, dx \quad \text{for every } j \in \mathbb{N}, \tag{31}$$

since $Pu_j \in \operatorname{span}\{e_{l+1}, \dots, e_m\}$ and $\operatorname{span}\{e_{l+1}, \dots, e_m\}$ is orthogonal to H_l and $H_m^\perp$ also in $L^2(\Omega)$.

Therefore, by Remark 4.2, (29), (30), (31) and (15) we have that

$$\begin{aligned} \langle Q\nabla\mathscr{F}_\lambda(u_j), u_j\rangle &= \langle \nabla\mathscr{F}_\lambda(u_j), u_j\rangle - \langle P\nabla\mathscr{F}_\lambda(u_j), u_j\rangle \\ &= \|u_j\|^2 - \lambda\int_\Omega u_j^2\, dx + \gamma\int_\Omega [(u_j+1)_-]^{p-1}\, u_j\, dx \\ &\quad - \langle P\left(u_j - \mathscr{L}_K^{-1}(\lambda u_j - \gamma\,[(u_j+1)_-]^{p-1})\right), u_j\rangle \\ &= \|u_j\|^2 - \lambda\int_\Omega u_j^2\, dx + \gamma\int_\Omega [(u_j+1)_-]^{p-1}\, u_j\, dx - \|Pu_j\|^2 \\ &\quad + \lambda\langle Pu_j, \mathscr{L}_K^{-1}u_j\rangle - \gamma\langle Pu_j, \mathscr{L}_K^{-1}\left([(u_j+1)_-]^{p-1}\right)\rangle \\ &= 2\mathscr{F}_\lambda(u_j) + \gamma\left(\frac{2}{p} - 1\right)\int_\Omega [(u_j+1)_-]^p\, dx - \gamma\int_\Omega [(u_j+1)_-]^{p-1}\, dx \\ &\quad - \|Pu_j\|^2 + \lambda\int_\Omega (Pu_j)^2\, dx - \gamma\int_\Omega [(u_j+1)_-]^{p-1}\, Pu_j\, dx. \end{aligned} \tag{32}$$

Observe that, by the Holder inequality and Lemma 2.1, we have that

$$\begin{aligned} \left|\int_\Omega [(u_j+1)_-]^{p-1}\, Pu_j\, dx\right| &\leq \left(\int_\Omega [(u_j+1)_-]^p\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |Pu_j|^p\, dx\right)^{\frac{1}{p}} \\ &\leq C\left(\int_\Omega [(u_j+1)_-]^p\, dx\right)^{\frac{p-1}{p}} \|Pu_j\|. \end{aligned} \tag{33}$$

On the other hand, by (32), we have that

$$\begin{aligned}\frac{\langle Q\nabla \mathscr{F}_\lambda(u_j), u_j\rangle}{\|u_j\|^{p/(p-1)}} = {} & \frac{2\mathscr{F}_\lambda(u_j)}{\|u_j\|^{p/(p-1)}} + \gamma\left(\frac{2}{p}-1\right)\frac{\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^{p/(p-1)}}\\ & - \frac{\gamma\int_\Omega [(u_j+1)_-]^{p-1}\,dx}{\|u_j\|^{p/(p-1)}} - \frac{\|Pu_j\|^2}{\|u_j\|^{p/(p-1)}} - \frac{\lambda\int_\Omega (Pu_j)^2\,dx}{\|u_j\|^{p/(p-1)}}\\ & - \frac{\gamma\int_\Omega [(u_j+1)_-]^{p-1}\,Pu_j\,dx}{\|u_j\|^{p/(p-1)}},\end{aligned}$$

and thus, by (i), (ii) and (iii) and throwing away the non-positive terms, we get

$$\begin{aligned}0 &\le \gamma\left(1-\frac{2}{p}\right)\frac{\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^{p/(p-1)}}\\ &= -\frac{\langle Q\nabla \mathscr{F}_\lambda(u_j), u_j\rangle}{\|u_j\|^{p/(p-1)}} + \frac{2\mathscr{F}_\lambda(u_j)}{\|u_j\|^{p/(p-1)}} - \frac{\gamma\int_\Omega [(u_j+1)_-]^{p-1}\,dx}{\|u_j\|^{p/(p-1)}}\\ &\quad - \frac{\|Pu_j\|^2}{\|u_j\|^{p/(p-1)}} - \frac{\lambda\int_\Omega (Pu_j)^2\,dx}{\|u_j\|^{p/(p-1)}} - \frac{\gamma\int_\Omega [(u_j+1)_-]^{p-1}\,Pu_j\,dx}{\|u_j\|^{p/(p-1)}}\\ &\le -\frac{\gamma\int_\Omega [(u_j+1)_-]^{p-1}\,Pu_j\,dx}{\|u_j\|^{p/(p-1)}} + o(1),\end{aligned}$$

with $o(1) \to 0$ as $j \to \infty$. Therefore, using the Holder inequality and Lemma 2.1, we get that

$$\begin{aligned}0 &\le \gamma\left(1-\frac{2}{p}\right)\frac{\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^{p/(p-1)}}\\ &\le a_1\left(\frac{\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^{p/(p-1)}}\right)^{1-\frac{1}{p}}\frac{\|Pu_j\|}{\|u_j\|^{1/(p-1)}} + o(1),\end{aligned}\tag{34}$$

with $a_1 > 0$ and $o(1) \to 0$ as $j \to \infty$. Now, since $\|Pu_j\| \to 0$, see (ii), and $\|u_j\| \to \infty$ as $j \to \infty$, the previous inequality implies that there exists $a_2 > 0$ such that

$$M(u_j) \doteq \frac{\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^{p/(p-1)}} \le a_2 \quad \text{for every } j \in \mathbb{N}.\tag{35}$$

Thus, we can suppose that $M(u_j) \to M$ as $j \to \infty$. Therefore, passing to the limit as $j \to \infty$ in (34), we have that, for every $\varepsilon > 0$

$$0 \le M \le \varepsilon M^{1-\frac{1}{p}},$$

which implies that $M = 0$. In this way, we get that

$$M(u_j) = \frac{\displaystyle\int_\Omega [(u_j+1)_-]^p \, dx}{\|u_j\|^{p/(p-1)}} \underset{j\to\infty}{\longrightarrow} 0,$$

and thus

$$\frac{[(u_j+1)_-]^{p-1}}{\|u_j\|} \to 0 \quad \text{in } L^{p/(p-1)}(\Omega) = L^{p'}(\Omega). \tag{36}$$

Then, since $\mathscr{L}_K^{-1} : L^{p'}(\Omega) \to X_0^s$ is a compact operator, by (27) and (36), we have that

$$\begin{aligned} \frac{Q\mathscr{L}_K^{-1}\left(\lambda u_j - \gamma\,[(u_j+1)_-]^{p-1}\right)}{\|u_j\|} &= Q\mathscr{L}_K^{-1}\left(\lambda\frac{u_j}{\|u_j\|} - \gamma\frac{[(u_j+1)_-]^{p-1}}{\|u_j\|}\right) \\ &\underset{j\to\infty}{\longrightarrow} \lambda Q\mathscr{L}_K^{-1}u. \end{aligned} \tag{37}$$

Moreover, by Remark 4.2, (29) and (iii) we get that

$$\frac{Q\nabla\mathscr{F}_\lambda(u_j)}{\|u_j\|} = \frac{u_j}{\|u_j\|} - \frac{Pu_j}{\|u_j\|} - Q\mathscr{L}_K^{-1}\left(\lambda\frac{u_j}{\|u_j\|} - \gamma\frac{[(u_j+1)_-]^{p-1}}{\|u_j\|}\right) \underset{j\to\infty}{\longrightarrow} 0,$$

and thus, by (27), (ii) and (37)

$$\frac{u_j}{\|u_j\|} \to \lambda Q\mathscr{L}_K^{-1}u = u \quad \text{in } X_0^s, \tag{38}$$

with $u \in H_l \oplus H_m^\perp$ and $\|u\| = 1$.

Therefore, for every $v \in X_0^s$, considering (ii), (36), (38) and the fact that $\mathscr{L}_K^{-1}$ is compact, we have that

$$\begin{aligned} \frac{Q\mathscr{F}'_\lambda(u_j)v}{\|u_j\|} &= \frac{\langle Qu_j, v\rangle}{\|u_j\|} - \frac{\langle Q\mathscr{L}_K^{-1}\left(\lambda u_j - \gamma\,[(u_j+1)_-]^{p-1}\right), v\rangle}{\|u_j\|} \\ &= \left\langle \frac{u_j}{\|u_j\|}, v\right\rangle - \frac{\langle Pu_j, v\rangle}{\|u_j\|} - \lambda\left\langle Q\mathscr{L}_K^{-1}\left(\frac{u_j}{\|u_j\|}\right), v\right\rangle \\ &\quad + \gamma\left\langle Q\mathscr{L}_K^{-1}\left(\frac{[(u_j+1)_-]^{p-1}}{\|u_j\|}\right), v\right\rangle \\ &\underset{j\to\infty}{\longrightarrow} \langle u, v\rangle - \lambda\langle Q\mathscr{L}_K^{-1}u, v\rangle, \end{aligned}$$

and thus, by (15) and the fact that $Qu = u$, since $u \in H_l \oplus H_m^\perp$, we have that

$$\begin{aligned} 0 &= \langle u, v\rangle - \lambda \langle \mathscr{L}_K^{-1} u, Qv\rangle = \langle u, v\rangle - \lambda \int_\Omega uQv\,dx \\ &= \langle u, v\rangle - \lambda \int_\Omega Quv\,dx \\ &= \langle u, v\rangle - \lambda \int_\Omega uv\,dx, \end{aligned}$$

for every $v \in X_0^s$.

This means that u is a nontrivial ($\|u\| = 1$) and non-negative (see (28)) eigenfunction in the space $H_l \oplus H_m^\perp$, but this is a contradiction since $\lambda \neq \lambda_1$. $\square$

Now we are ready to prove the (∇)-condition for $\mathscr{F}_\lambda$.

Proposition 4.9 ((∇)-condition for $\mathscr{F}_\lambda$)**.** *Let $l, m \in \mathbb{N}$, with $l \geq 1$ and $m \geq l + 1$, be such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$.*

Then, for every $\sigma > 0$ there exists $\varepsilon_\sigma > 0$ such that, for every $\lambda \in [\lambda_l + \sigma, \lambda_{m+1} - \sigma]$ and for every $\varepsilon', \varepsilon'' \in (0, \varepsilon_\sigma)$, with $\varepsilon' < \varepsilon''$, functional $\mathscr{F}_\lambda$ satisfies $(\nabla)\left(\mathscr{F}_\lambda, H_l \oplus H_m^\perp, \varepsilon', \varepsilon''\right)$-condition.

Proof. Let P, Q denote the orthogonal projections introduced in Lemma 4.8.

We proceed by contradiction, supposing that there exists $\sigma > 0$ such that, for every $\varepsilon_0 > 0$ there exist $\bar\lambda \in [\lambda_l + \sigma, \lambda_{m+1} - \sigma] \setminus \{\lambda_1\}$ and $\varepsilon', \varepsilon'' \in (0, \varepsilon_0)$, with $\varepsilon' < \varepsilon''$, such that

$$(\nabla)\left(\mathscr{F}_{\bar\lambda}, H_l \oplus H_m^\perp, \varepsilon', \varepsilon''\right) \text{ does not hold.} \tag{39}$$

Choose as ε_0 the one provided in Lemma 4.7 (associated to σ).

As a consequence of (39), there exists a sequence $(u_j)_j \subseteq X_0^s$ such that

$$\begin{aligned} &\mathscr{F}_{\bar\lambda}(u_j) \in [\varepsilon', \varepsilon''] \text{ for every } j \in \mathbb{N}; \\ &\operatorname{dist}\left(u_j, H_l \oplus H_m^\perp\right) \underset{j\to\infty}{\longrightarrow} 0; \\ &Q\nabla\mathscr{F}_{\bar\lambda}(u_j) \underset{j\to\infty}{\longrightarrow} 0 \text{ in } X_0^s. \end{aligned} \tag{40}$$

Then, by Lemma 4.8, $(u_j)_j$ is bounded in X_0^s and thus, up to a subsequence (see Lemma 2.1)

$$\begin{aligned} &u_j \rightharpoonup u \text{ in } X_0^s \\ &u_j \to u \text{ in } L^q(\Omega), \text{ for every } q \in [1, 2^\star) \\ &u_j \to u \text{ a.e. in } \Omega. \end{aligned} \tag{41}$$

On the other hand, by Remark 4.2, we can write

$$\begin{aligned} Q\nabla\mathscr{F}_{\bar\lambda}(u_j) &= Qu_j - Q\mathscr{L}_K^{-1}\left(\bar\lambda u_j - \gamma\left[(u_j+1)_-\right]^{p-1}\right) \\ &= u_j - Pu_j - Q\mathscr{L}_K^{-1}\left(\bar\lambda - \gamma\left[(u_j+1)_-\right]^{p-1}\right). \end{aligned}$$

Moreover, by (41) and the Generalized Lebesgue Theorem,

$$\int_\Omega [(u_j+1)_-]^{\frac{(p-1)p}{p-1}}\,dx = \int_\Omega [(u_j+1)_-]^p\,dx \underset{j\to\infty}{\longrightarrow} \int_\Omega [(u+1)_-]^p\,dx,$$

and so

$$[(u_j+1)_-]^{p-1} \to [(u+1)_-]^{p-1} \text{ in } L^{p'}(\Omega). \tag{42}$$

Therefore, since $\mathscr{L}_K^{-1} : L^{p'}(\Omega) \to X_0^s$ is a compact operator, by (42) and (41) we have that

$$Q\mathscr{L}_K^{-1}\left(\bar\lambda u_j - \gamma\,[(u_j+1)_-]^{p-1}\right) \underset{j\to\infty}{\longrightarrow} Q\mathscr{L}_K^{-1}\left(\bar\lambda u - \gamma\,[(u+1)_-]^{p-1}\right). \tag{43}$$

Now, by (40), we get

$$Q\nabla\mathscr{F}_{\bar\lambda}(u_j) = u_j - Pu_j - Q\mathscr{L}_K^{-1}\left(\bar\lambda u_j - \gamma\,[(u_j+1)_-]^{p-1}\right) \underset{j\to\infty}{\longrightarrow} 0,$$

and hence

$$u_j \to Q\mathscr{L}_K^{-1}\left(\bar\lambda u - \gamma\,[(u+1)_-]^{p-1}\right) \doteq u \text{ in } X_0^s. \tag{44}$$

Now, again from (40),

$$\mathscr{F}'_{\bar\lambda}(u_j)\varphi = \langle u_j, \varphi\rangle - \bar\lambda\int_\Omega u_j\varphi\,dx + \gamma\int_\Omega [(u_j+1)_-]^{p-1}\,\varphi\,dx \underset{j\to\infty}{\longrightarrow} 0,$$

for every $\varphi \in H_l \oplus H_m^\perp$. But, on the other hand, by (44), (41) and the Generalized Lebsegue Theorem,

$$\langle u_j, \varphi\rangle \underset{j\to\infty}{\longrightarrow} \langle u, \varphi\rangle$$
$$\int_\Omega u_j\varphi\,dx \underset{j\to\infty}{\longrightarrow} \int_\Omega u\varphi\,dx$$
$$\int_\Omega [(u_j+1)_-]^{p-1}\,\varphi\,dx \underset{j\to\infty}{\longrightarrow} \int_\Omega [(u+1)_-]^{p-1}\,\varphi\,dx,$$

for every $\varphi \in H_l \oplus H_m^\perp$, and thus u is a critical point of $\mathscr{F}_{\bar\lambda}$, constrained on $H_l \oplus H_m^\perp$.

Then, by Lemma 4.7, $u \equiv 0$ and hence $\mathscr{F}_{\bar\lambda}(u) = 0$.

But, by (40), $0 < \varepsilon' \le \mathscr{F}_{\bar\lambda}(u_j)$ for every $j \in \mathbb{N}$ and thus, since $\mathscr{F}_{\bar\lambda}$ is obviously continuous and (44) holds, a contradiction arises. □

4.3 The multiplicity result

First of all, we want to produce an existence result for problem (P_λ) using Theorem 2.5. In order to achieve this result, we need to prove the following two lemmas.

Lemma 4.10. *Let $l, m \in \mathbb{N}$, with $l \geq 1$ and $m \geq l+1$, be such that $\lambda_l < \lambda < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$, with $\lambda > \lambda_m^*$.*

Then

$$\lim_{\substack{\|u\| \to \infty, \\ u \in H_m}} \mathscr{F}_\lambda(u) = -\infty.$$

Proof. By contradiction, suppose that there exists a sequence $(u_j)_j \subseteq H_m$ and a constant $M \in \mathbb{R}$ such that $\|u_j\| \to \infty$ as $j \to \infty$ and

$$\mathscr{F}_\lambda(u_j) \geq M, \quad \text{for every } j \in \mathbb{N}. \tag{45}$$

Therefore, by Lemma 2.1 and the fact that H_m is finite-dimensional,

$$\frac{u_j}{\|u_j\|} \to u \text{ in } X_0^s, \text{ in } L^q(\Omega) \text{ and a.e. in } \Omega, \tag{46}$$

for every $q \in [1, 2^\star)$ and $\|u\| = 1$.

Now, dividing both sides of (45) by $\|u_j\|^p$, by (46) we get that

$$\begin{aligned} o(1) = \frac{M}{\|u_j\|^p} \leq \frac{\mathscr{F}_\lambda(u_j)}{\|u_j\|^p} &= \frac{1}{2}\|u_j\|^{2-p} - \frac{\lambda}{2}\|u_j\|^{2-p} \int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega (u_-)^p\,dx + o(1) \\ &\underset{j \to \infty}{\longrightarrow} -\frac{\gamma}{p}\int_\Omega (u_-)^p\,dx, \end{aligned}$$

with $o(1) \to$ as $j \to \infty$, and thus

$$u \geq 0 \text{ in } \Omega. \tag{47}$$

Therefore, by (45), (46), (47), Remark 4.4 and the fact that $\lambda > \lambda_m^*$, we have that

$$\begin{aligned} o(1) = \frac{M}{\|u_j\|^p} \leq \frac{\mathscr{F}_\lambda(u_j)}{\|u_j\|^2} &= \frac{1}{2} - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{\gamma}{p}\lim_{j \to \infty} \frac{\displaystyle\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^2} + o(1) \\ &\leq \frac{1}{2} - \frac{\lambda}{2}\int_\Omega u^2\,dx \leq \frac{1}{2}(\lambda_m^* - \lambda)\int_\Omega u^2\,dx < 0, \end{aligned}$$

which is clearly absurd. □

Lemma 4.11. *Let $l, m \in \mathbb{N}$, with $l \geq 1$ and $m \geq l+1$, be such that $\lambda_l < \lambda < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$, with $\lambda > \lambda_m^*$.*

Then

$$\lim_{\lambda \to \lambda_{l+1}} \sup_{u \in H_m} \mathscr{F}_\lambda(u) = 0.$$

Proof. Suppose by contradiction that there exist $(\mu_j)_j \subseteq \mathbb{R}^+$ such that $\mu_j \to \lambda_{l+1}$ as $j \to \infty$, a sequence $(u_j)_j \subseteq H_m$ and $\varepsilon > 0$ such that for every $j \in \mathbb{N}$

$$\mathscr{F}_{\mu_j}(u_j) = \sup_{u \in H_m} \mathscr{F}_{\mu_j}(u) \geq \varepsilon > 0. \tag{48}$$

Let us remark that $\mathscr{F}_{\mu_j}$ attains a maximum in H_m, by Lemma 4.10 and the Weierstrass Theorem.

Now, two possibilities occur.

If $(u_j)_j$ is bounded in X_0^s, then, by Lemma 2.1 and the fact that H_m is finite dimensional,

$$\begin{aligned} u_j &\to u \quad \text{in } X_0^s \\ u_j &\to u \quad \text{in } L^q(\Omega), \text{ for every } q \in [1, 2^\star) \\ u_j &\to u \quad \text{a.e. in } \Omega. \end{aligned} \tag{49}$$

Using (49) and the assumptions, we get that

$$\mathscr{F}_{\mu_j}(u_j) \underset{j\to\infty}{\longrightarrow} \mathscr{F}_{\lambda_{l+1}}(u).$$

Therefore, by (48) and (7),

$$\begin{aligned} 0 < \varepsilon \le \mathscr{F}_{\lambda_{l+1}}(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda_{l+1}}{2}\int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\,dx \\ &\le \frac{1}{2}(\lambda_m - \lambda_{l+1})\int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\,dx \le 0, \end{aligned}$$

which is a contradiction.

Conversely, if $(u_j)_j$ is unbounded in X_0^s, then we can assume that there exists $w \in H_m$ such that

$$\frac{u_j}{\|u_j\|} \to w \text{ in } X_0^s, \text{ in } L^q(\Omega) \text{ and a.e. in } \Omega, \tag{50}$$

for every $q \in [1, 2^\star)$ (see Lemma 2.1), with $\|w\| = 1$.

Dividing both sides of (48) by $\|u_j\|^p$, we get that

$$\begin{aligned} 0 < \frac{\mathscr{F}_{\mu_j}(u_j)}{\|u_j\|^p} &= \frac{1}{2}\|u_j\|^{2-p} - \frac{\mu_j}{2}\|u_j\|^{2-p}\int_\Omega \left(\frac{u_j}{\|u_j\|}\right)^2 dx \\ &\quad - \frac{\gamma}{p}\int_\Omega \left[\left(\frac{u_j}{\|u_j\|} + \frac{1}{\|u_j\|}\right)_-\right]^p dx \\ &= o(1) - \frac{\gamma}{p}\int_\Omega (w_-)^p\,dx, \end{aligned}$$

and thus $w \ge 0$ in Ω.

Therefore, again by (48), now diving by $\|u_j\|^2$, we have that

$$0 < \frac{1}{2} - \frac{\mu_j}{2}\int_\Omega \left(\frac{u_j}{\|u_j\|}\right)^2 dx - \frac{\gamma}{p}\frac{\displaystyle\int_\Omega [(u_j+1)_-]^p\,dx}{\|u_j\|^2},$$

and so, passing to the limit as $j \to \infty$,

$$0 \le \frac{1}{2} - \frac{\lambda_{l+1}}{2}\int_\Omega w^2\,dx = \mathscr{F}_{\lambda_{l+1}}(w),$$

but

$$0 \le \mathscr{F}_{\lambda_{l+1}}(w) = \frac{1}{2} - \frac{\lambda_{l+1}}{2} \int_\Omega w^2\,dx \le \frac{1}{2}(\lambda_m^* - \lambda) \int_\Omega w^2\,dx < 0,$$

which is impossible. □

Theorem 4.12 (Existence via the (∇)-Theorem)**.** *Let $l, m \in \mathbb{N}$, with $l \ge 1$ and $m \ge l+1$, be such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$.*

Then, there exists $\delta > 0$ such that, for every $\lambda \in (\lambda_{l+1} - \delta, \lambda_{l+1})$, with $\lambda_{l+1} - \delta \ge \lambda_m^*$, (P_λ) admits two nontrivial solutions $u_1, u_2 \in X_0^s$ such that

$$0 < \mathscr{F}_\lambda(u_i) \le \sup_{u \in H_m} \mathscr{F}_\lambda(u), \ \textit{for every } i = 1, 2.$$

Proof. Fix $\sigma > 0$ and choose ε_σ provided in Proposition 4.9. Then, for every $\lambda \in [\lambda_l + \sigma, \lambda_{m+1} - \sigma] \setminus \{\lambda_1\}$ and for every $\varepsilon', \varepsilon'' \in (0, \varepsilon_\sigma)$, $\mathscr{F}_\lambda$ satisfies the $(\nabla)(\mathscr{F}_\lambda, H_l \oplus H_m^\perp, \varepsilon', \varepsilon'')$-condition.

Since $\lambda < \lambda_{l+1}$, by Proposition 4.6, the geometry condition of Theorem 2.5 holds true, with

$$\sup \mathscr{F}_\lambda(\Delta) \le \sup_{u \in H_m} \mathscr{F}_\lambda(u),$$

where $\Delta \doteq \{u \in X_1 \oplus X_2 : \|u\| \le R\}$.

By Lemma 4.11, there exists $\delta \le \sigma$ such that, if $\lambda \in (\lambda_{l+1} - \delta, \lambda_{l+1})$ then

$$\sup_{u \in H_m} \mathscr{F}_\lambda(u) < \varepsilon''.$$

Moreover, we saw in the proof of Theorem 1.2 that $\mathscr{F}_\lambda$ satisfies the (PS)-condition for every $\lambda \in \mathbb{R} \setminus \{\lambda_1\}$.

Then, by Theorem 2.5, there exist u_1, u_2 critical points of $\mathscr{F}_\lambda$ such that $\mathscr{F}_\lambda(u_i) \in [\varepsilon', \varepsilon'']$, for every $i = 1, 2$.

Since ε'' is any number smaller than ε_σ and larger than $\sup \mathscr{F}_\lambda(\Delta)$ (see the notation of Theorem 2.5), we get that

$$\mathscr{F}_\lambda(u_i) \le \sup \mathscr{F}_\lambda(\Delta) \le \sup_{u \in H_m} \mathscr{F}_\lambda(u),$$

for every $i = 1, 2$. □

Remark 4.13. Of course, the existence of two solutions for λ near λ_l is obvious from bifurcation theory, but the application of the ∇-Theorem gives us precise estimates on the associated critical values, which are fundamental to prove our main theorem below, which is a precise formulation of Theorem 1.5.

Theorem 4.14. *Let $l, m \in \mathbb{N}$, with $l \ge 1$ and $m \ge l+1$, be such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_m < \lambda_{m+1}$.*

Then, there exists $\delta > 0$ such that, for every $\lambda \in (\lambda_{l+1} - \delta, \lambda_{l+1})$, with $\lambda_m^* \le \lambda_{l+1} - \delta$, (P_λ) admits three nontrivial solutions.

Proof. As a consequence of Theorem 4.12, there exists $\bar{\delta} > 0$ such that, for every $\lambda \in (\lambda_{l+1} - \bar{\delta}, \lambda_{l+1})$, with $\lambda_{l+1} - \bar{\delta} \geq \lambda_m^*$, (P_λ) has two nontrivial solutions $u_1, u_2 \in X_0^s$, verifying

$$0 < \mathscr{F}_\lambda(u_i) \leq \varepsilon'', \tag{51}$$

see above.

Since $\lambda \neq \lambda_1$ and $\lambda < \lambda_m$, we can use Theorem 1.2 to find another solution $u_3 \in X_0^s$, adopting the decomposition $X_0^s = H_m \oplus H_m^\perp$.

Indeed, for every $\tau > 0$, for $u \in H_m^\perp$ with $\|u\| = \rho > 0$ small enough, using (6), Lemma 3.1 and Lemma 2.1, we have that

$$\begin{aligned}
\mathscr{F}_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\;dx \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{m+1}}\right)\|u\|^2 - \tau C\|u\|^p \\
&= \rho^2\left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{m+1}}\right) - \tau C\rho^{p-2}\right).
\end{aligned}$$

Therefore, choosing ρ small enough, we get

$$\inf_{\substack{u \in H_m^\perp,\\ \|u\| = \rho}} \mathscr{F}_\lambda(u) := \alpha > 0. \tag{52}$$

On the other hand, if $u \in H_m$, by (7) we get that

$$\mathscr{F}_\lambda(u) \leq \frac{1}{2}(\lambda_m - \lambda)\int_\Omega u^2\,dx - \frac{\gamma}{p}\int_\Omega [(u+1)_-]^p\;dx \leq 0,$$

since $\lambda > \lambda_m$. Moreover, taking $v = u + te_{m+1}$, with $u \in H_m$ and $t > 0$, since u and e_{m+1} are orthogonal in $L^2(\Omega)$ and in X_0^s (see Remark 1.3), again by (6), we get that

$$\begin{aligned}
\mathscr{F}_\lambda(v) &= \mathscr{F}_\lambda(u + te_{m+1}) \\
&= \frac{1}{2}\|u\|^2 + \frac{t^2}{2}\|e_{m+1}\|^2 - \frac{\lambda}{2}\int_\Omega u^2\,dx - \lambda\frac{t^2}{2}\int_\Omega e_{m+1}^2\,dx \\
&\quad - \frac{\gamma}{p}\int_\Omega [(u + te_{m+1} + 1)_-]^p\;dx \\
&\leq \frac{1}{2}(\lambda_m - \lambda)\int_\Omega u^2\,dx + \frac{t^2}{2}(\lambda_{m+1} - \lambda)\int_\Omega e_{m+1}^2\,dx \\
&\quad - \frac{\gamma}{p}\int_\Omega [(u + te_{m+1} + 1)_-]^p\;dx \\
&\leq \frac{t^2}{2}(\lambda_{m+1} - \lambda) - \gamma\frac{t^p}{p}\int_\Omega \left[\left(\frac{u}{t} + e_{m+1} + \frac{1}{t}\right)_-\right]^p\,dx,
\end{aligned}$$

and thus, using the Generalized Lebesgue Theorem and the fact that $(e_{m+1})_- \not\equiv 0$, we get that $\mathscr{F}_\lambda(v) \to -\infty$ as $t \to \infty$.

Then, the geometric condition of the classical Linking Theorem holds and, since we have already proved the $(PS)_c$-condition for $\mathscr{F}_\lambda$ for every $\lambda \neq \lambda_1$ (see the proof of Theorem 1.2), we find another solution u_3 for problem (P_λ) with the property that

$$\mathscr{F}_\lambda(u_3) \geq \inf_{\substack{u \in H_m^\perp, \\ \|u\|=\rho}} \mathscr{F}_\lambda(u) = \alpha. \tag{53}$$

Therefore, by Lemma 4.11, (51) and (53), there exists $\delta \leq \bar{\delta}$ such that, for every $\lambda \in (\lambda_{l+1} - \delta, \lambda_{l+1})$,

$$\varepsilon'' < \alpha = \inf_{\substack{u \in H_m^\perp, \\ \|u\|=\rho}} \mathscr{F}_\lambda(u),$$

and thus

$$0 < \mathscr{F}_\lambda(u_i) < \mathscr{F}_\lambda(u_3), \quad \text{for every } i = 1, 2,$$

finally finding the announced third nontrivial solution. □

5 $\gamma \to \infty$

In order to underline the dependence of the functional and of the solutions from γ, in this section we will use the following replacements:

$$(P_\lambda) \rightsquigarrow (P_\lambda^\gamma), \;\; \mathscr{F}_\lambda \rightsquigarrow \mathscr{F}_\lambda^\gamma \text{ and } u_i \rightsquigarrow u_i^\gamma, \; i = 1, 2, 3.$$

In this way, we notice all the solutions we found in Theorem 4.14 enjoy the property that for every $\gamma > 0$.

$$0 < \mathscr{F}_\lambda^\gamma(u_1^\gamma), \mathscr{F}_\lambda^\gamma(u_2^\gamma) < \mathscr{F}_\lambda^\gamma(u_3^\gamma) \leq \sup \mathscr{F}_\lambda^\gamma(H_{m+1}) \leq \sup \mathscr{F}_\lambda^0(H_{m+1}) \tag{54}$$

with $\sup \mathscr{F}_\lambda^0(H_{m+1}) < \infty$ by Lemma 4.10, since $\lambda < \lambda_{m+1}$.

Moreover, also in Theorem 1.2 we have a similar uniform estimate. Indeed, if $\lambda < \lambda_1$ we got the existence of a nontrivial solution u_γ via the Mountain Pass Theorem (see the proof of Theorem 1.2. Hence, we have the following information:

$$\mathscr{F}_\lambda^\gamma(u_\gamma) = \inf_{\varphi \in \Gamma} \max_{t \in [0,1]} \mathscr{F}_\lambda^\gamma(\varphi(t)),$$

for every $\gamma > 0$, where $\Gamma = \{\varphi \in \mathscr{C}\big([0,1], X_0^s\big) : \varphi(0) = 0, \varphi(1) = e\}$. In this way, choosing $e = -Re_1$ for some $R > 0$ large enough and taking the path $\varphi(t) = -tRe_1$, $t \in [0,1]$, we get that

$$\mathscr{F}_\lambda^\gamma(-tRe_1) = \lambda_1 R^2 \frac{t^2}{2} - \lambda R^2 \frac{t^2}{2} - \frac{\gamma}{p} \int_\Omega [(-te_1 + 1)_-]^p \, dx \leq \frac{t^2 R^2}{2}(\lambda_1 - \lambda)$$

and hence

$$\sup_{\gamma>0} \mathscr{F}_\lambda^\gamma(u_\gamma) \le \frac{\lambda_1-\lambda}{2} R^2. \tag{55}$$

On the other hand, if $\lambda > \lambda_1$, and so $\lambda_i \le \lambda < \lambda_{i+1}$ for some $i \in \mathbb{N}$, a nontrivial solution is obtained using the Linking Theorem with the decomposition $X_0^s = H - i \oplus H_i^\perp$. In particular, if we choose $e = e_{i+1}$ and $R > \rho > 0$ (see Theorem 2.3), every nontrivial critical point u_γ of $\mathscr{F}_\lambda^\gamma$ satisfies

$$\mathscr{F}_\lambda^\gamma(u_\gamma) = \inf_{h\in\mathscr{H}} \max_{v\in Q} \mathscr{F}_\lambda^\gamma\big(h(v)\big),$$

where

$$Q = \{v = u + te_{i+1} \,\big|\, u \in H_i,\ \|u\| \le R,\ t \in (0,R)\}$$

and

$$\mathscr{H} = \{h \in \mathscr{C}(\overline{Q}, X_0^s) \,:\, h_{|\partial Q} = Id\}.$$

Taking $h = Id_Q$ and $v \in Q$, by the same estimates provided in the proof of Theorem 1.2, we have that

$$\begin{aligned}
\mathscr{F}_\lambda^\gamma(h(v)) &= \mathscr{F}_\lambda^\gamma(u + te_{i+1}) \\
&\le \frac{1}{2}(\lambda_i - \lambda)\int_\Omega u^2\,dx + \frac{t^2}{2}(\lambda_{i+1}-\lambda) - \frac{\gamma}{p}\int_\Omega [(u+te_{i+1}+1)_-]^p\,dx \\
&\le \frac{R^2}{2}(\lambda_{i+1}-\lambda).
\end{aligned}$$

This being valid for every $\gamma > 0$, we get that

$$\sup_{\gamma>0} \mathscr{F}_\lambda^\gamma(u_\gamma) \le \frac{R^2}{2}(\lambda_{i+1}-\lambda). \tag{56}$$

Finally, if $\lambda = \lambda_1$, we have

$$\mathscr{F}_\lambda^\gamma(te_1) = 0$$

for every $\gamma, t > 0$.

Notice that in this last case it is clear that no upper bound is available for the set of solutions. On the other hand, in our next result we will show that all the solutions found in Theorem 1.2 for $\lambda \ne \lambda_1$ and in Theorem 1.5 are bounded thanks to the estimates found in (55), in (56) and in (54). More precisely, we have the following *a priori* estimate.

Theorem 5.1. *If $(u_\gamma)_{\gamma>0}$ is a family of solutions of (P_λ^γ) with*

$$\sup_\gamma \mathscr{F}_\lambda^\gamma(u_\gamma) < \infty$$

and $\lambda \ne \lambda_1$, then $(u_\gamma)_{\gamma>0}$ is bounded in X_0^s.

Proof. Case $\lambda < \lambda_1$. By the Poincaré inequality we immediately get that

$$\begin{aligned}
p\mathscr{F}_\lambda^\gamma(u_\gamma) &= p\mathscr{F}_\lambda^\gamma(u_\gamma) - (\mathscr{F}_\lambda^\gamma)'(u_\gamma)u_\gamma \\
&= \left(\frac{p}{2} - 1\right)\left(\|u_\gamma\|^2 - \lambda \int_\Omega u_\gamma^2\, dx\right) + \gamma \int_\Omega [(u_\gamma + 1)_-]^{p-1}\, dx \\
&\geq C\|u_\gamma\|^2,
\end{aligned}$$

with $C > 0$. Thus, by assumption, $(u_\gamma)_\gamma$ is bounded in X_0^s, as claimed.

Case $\lambda > \lambda_1$. Suppose by contradiction that $(u_\gamma)_\gamma$ is unbounded in X_0^s. Then, there exists a sequence $(u_n)_n \doteq (u_{\gamma_n})_n$ such that $\|u_n\| \to \infty$ as $n \to \infty$ and

$$\frac{u_n}{\|u_n\|} \rightharpoonup w \text{ in } X_0^s, \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{57}$$

We know that

$$\begin{aligned}
0 &= (\mathscr{F}_\lambda^{\gamma_n})'(u_n)u_n \\
&= 2\mathscr{F}_\lambda^{\gamma_n}(u_n) + \gamma_n\left(\frac{2}{p} - 1\right)\int_\Omega [(u_n + 1)_-]^p\, dx - \gamma_n \int_\Omega [(u_n + 1)_-]^{p-1}\, dx.
\end{aligned}$$

In particular, by assumption, we get the existence of $C > 0$ such that

$$\gamma_n\left(1 - \frac{2}{p}\right)\int_\Omega [(u_n + 1)_-]^p\, dx + \gamma_n \int_\Omega [(u_n + 1)_-]^{p-1}\, dx = 2\mathscr{F}_\lambda^{\gamma_n}(u_n) \leq C.$$

Therefore, dividing the previous inequality by $\|u_n\|$, we have that

$$\lim_{n\to\infty} \gamma_n \int_\Omega \frac{[(u_n + 1)_-]^p}{\|u_n\|}\, dx = \lim_{n\to\infty} \gamma_n \int_\Omega \frac{[(u_n + 1)_-]^{p-1}}{\|u_n\|}\, dx = 0,$$

and so

$$\lim_{n\to\infty} \gamma_n \int_\Omega \frac{[(u_n + 1)_-]^{p-1}\, u_n}{\|u_n\|}\, dx = 0. \tag{58}$$

Moreover, dividing $(\mathscr{F}_\lambda^{\gamma_n})'(u_n)u_n$ by $\|u_n\|^2$ and passing to the limit as $n \to \infty$, as a consequence of (58) and (57), we get that

$$\begin{aligned}
0 &= \frac{(\mathscr{F}_\lambda^{\gamma_n})'(u_n)u_n}{\|u_n\|^2} \\
&= 1 - \lambda \int_\Omega \frac{u_n^2}{\|u_n\|^2}\, dx + \gamma_n \int_\Omega \frac{[(u_n + 1)_-]^{p-1}\, u_n}{\|u_n\|^2}\, dx \\
&\underset{n\to\infty}{\longrightarrow} 1 - \lambda \int_\Omega w^2\, dx,
\end{aligned}$$

and hence

$$w \not\equiv 0. \tag{59}$$

Furthermore, again by (57) we have

$$\begin{aligned} 0 &= \frac{\mathscr{F}_\lambda^{\gamma_n}(u_n)}{\gamma_n \|u_n\|^p} \\ &= \frac{1}{2\gamma_n}\|u_n\|^{2-p} - \frac{\lambda}{2\gamma_n}\int_\Omega \frac{u_n^2}{\|u_n\|^p}\,dx - \frac{1}{p}\int_\Omega \frac{[(u_n+1)_-]^p}{\|u_n\|^p}\,dx \\ &\underset{n\to\infty}{\longrightarrow} -\frac{1}{p}\int_\Omega (w_-)^p\,dx, \end{aligned}$$

and so

$$w \geq 0 \ \text{ in } \Omega. \tag{60}$$

Choosing $r > 0$ such that $re_1 \leq 1$ a.e. in Ω, we have

$$0 = (\mathscr{F}_\lambda^{\gamma_n})'(u_n)re_1 \quad = r(\lambda_1 - \lambda)\int_\Omega u_n e_1\,dx + \gamma_n \int_\Omega [(u_n+1)_-]^{p-1}\,re_1\,dx. \tag{61}$$

We observe that

$$\begin{aligned} \int_\Omega [(u_n+1)_-]^{p-1}\,re_1\,dx &= \int_\Omega [(u_n+1)_-]^{p-1}\,(re_1 - 1 - u_n + 1 + u_n)\,dx \\ &\leq -\int_\Omega [(u_n+1)_-]^p\;dx - \int_\Omega [(u_n+1)_-]^{p-1}\,u_n\,dx \\ &\leq -\int_\Omega [(u_n+1)_-]^{p-1}\,u_n\,dx. \end{aligned}$$

Hence, by (58)

$$0 \leq \gamma_n \int_\Omega \frac{[(u_n+1)_-]^{p-1}\,re_1}{\|u_n\|}\,dx \leq -\gamma_n \int_\Omega \frac{[(u_n+1)_-]^{p-1}\,u_n}{\|u_n\|}\,dx \underset{n\to\infty}{\longrightarrow} 0,$$

and so

$$\lim_{n\to\infty} \gamma_n \int_\Omega \frac{[(u_n+1)_-]^{p-1}\,e_1}{\|u_n\|\,dx} = 0.$$

Therefore, starting from (61), we obtain that

$$\lim_{n\to\infty} (\lambda_1 - \lambda)\int_\Omega \frac{u_n e_1}{\|u_n\|}\,dx = (\lambda_1 - \lambda)\int_\Omega we_1\,dx = 0$$

and thus, by (60), $w \equiv 0$, which is in contradiction with (59).

Hence, $(u_\gamma)_{\gamma>0}$ is bounded in X_0^s. $\square$

We notice that, thanks to (54), (55) and (56), all the set of solutions we found in the previous sections are equibounded, and so we may assume that any sequence of solutions $(u_{\gamma_n})_n$ converges weakly in X_0^s as $n \to \infty$. From now on, we will write $(u_\gamma)_{\gamma>0}$ to denote any sequence $(u_{\gamma_n})_n$ and we will write $u_\gamma \rightharpoonup u$ as $\gamma \to \infty$ meaning that $u_{\gamma_n} \rightharpoonup u$ as $n \to \infty$.

Since u is a weak limit in X_0^s, we cannot consider any set defined by pointwise values of u. For instance, it would be natural to define the "contact set" as

$$\{x \in \Omega : u(x) = -1\},$$

which is a closed set if u is continuous, but, at this stage, we don't have any tool which lets us say that u has such a regularity. For this reason, we introduce the following sets:

$$\mathcal{S}_\gamma := \frac{\{x \in \Omega \,:\, u_\gamma(x) < -1\}}{\sim},$$

where two sublevel $\mathcal{S}_\gamma$ and $\mathcal{S}'_\gamma$ are equivalent according to $\sim$ if they differ for a set of measure zero. Then, we define the "free" set

$$\mathcal{F} = \Big\{x \in \Omega \,:\text{there exists a neighborhood } U \text{ of } x \text{ and } \gamma_0 > 0 \text{ such that}$$
$$|U \cap \mathcal{S}_\gamma| = 0 \text{ for all } \gamma \geq \gamma_0\Big\},$$

where $|A|$ here stands for the Lebesgue measure of a set A. Of course, $\mathcal{F}$ is an open subset of Ω. Thus, $\mathfrak{C} := \Omega \setminus \mathcal{F}$ is closed, where

$$\mathfrak{C} = \Big\{x \in \Omega \,:\text{for every neighborhood } U \text{ of } x \text{ and all } \gamma_0 > 0$$
$$\text{there exists } \gamma \geq \gamma_0 \text{ such that } |U \cap \mathcal{S}_\gamma| > 0\Big\}.$$

We now show that in our situation, $\mathfrak{C}$ plays the role of the contact set in the standard obstacle problem. However, being here the sign of the inequality reversed with respect to the obstacle case, this situation is related to bounce problems, see [10], [17].

Theorem 5.2. *Let u_γ be a solution of (P_λ^γ) and suppose that $u_\gamma \rightharpoonup u$ in X_0^s as $\gamma \to \infty$. Then*

(i) *$u \geq -1$ a.e. in Ω;*

(ii) *there exists a positive Radon measure μ supported in the set $\mathfrak{C}$ such that*

$$\iint_{\mathscr{O}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)K(x-y)\,dx\,dy - \lambda \int_\Omega uv\,dx = -\int_\Omega v\,d\mu$$

for every $v \in \mathscr{C}_c^\infty(\Omega)$.

Proof. (i) Suppose by contradiction that there exists a set $\omega \subseteq \Omega$, with $|\omega| > 0$, such that $u + 1 < 0$ a.e. in ω. In this way, testing (P_λ^γ) with e_1,

$$(\lambda_1 - \lambda)\int_\Omega u_\gamma e_1\,dx = -\gamma \int_\Omega [(u_\gamma + 1)_-]^{p-1}\, e_1\,dx, \tag{62}$$

and passing to the limit as $\gamma \to \infty$, we would reach a contradiction.

(ii) Define the family of distributions $T_\gamma : \mathscr{C}_c^\infty(\Omega) \to \mathbb{R}$ as

$$\begin{aligned} T_\gamma(v) &\doteq \gamma \int_\Omega [(u_\gamma + 1)_-]^{p-1}\, v\, dx \\ &= -\left(\iint_{\mathscr{O}} \big(u_\gamma(x) - u_\gamma(y)\big)\big(v(x) - v(y)\big) K(x-y)\, dx\, dy - \lambda \int_\Omega u_\gamma v\, dx \right). \end{aligned}$$

Since by assumption $u_\gamma \rightharpoonup u$ in X_0^s, we have that $T_\gamma \to T$ in the sense of distributions, where $T : \mathscr{C}_c^\infty(\Omega) \to \mathbb{R}$ is defined as

$$T(v) \doteq -\left(\iint_{\mathscr{O}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) K(x-y)\, dx\, dy - \lambda \int_\Omega uv\, dx \right).$$

Now, it is readily seen that $T\gamma$ are positive distributions, and so $T(v) \geq 0$, as well. Thus, by the Riesz Representation Theorem, we get that there exists a positive Radon measure μ such that

$$T(v) = \int_\Omega v\, d\mu,$$

for every $v \in \mathscr{C}_c^\infty(\Omega)$.

Finally, we prove that $\operatorname{supp} \mu \subseteq \mathfrak{C}$: if $x_0 \in \Omega \setminus \mathfrak{C}$, then there exists a neighborhood U of x_0 and $\gamma_0 > 0$ such that $u_\gamma + 1 \geq 0$ a.e. in U for every $\gamma \geq \gamma_0$. Now, take $\phi \in \mathscr{C}_c^\infty(U)$, so that

$$\iint_{\mathscr{O}} \big(u_\gamma(x) - u_\gamma(y)\big)\big(\phi(x) - \phi(y)\big) K(x-y)\, dx\, dy - \lambda \int_\Omega u_\gamma \phi\, dx = 0$$

for every $\gamma \geq \gamma_0$. Passing to the limit as $\gamma \to \infty$, we get the claim. □

Acknowledgments

The authors wish to thank the anonymous referees for carefully reading the paper and for suggesting some improvements which improved its presentation.

References

[1] M. Allen, *A fractional free boundary problem related to a plasma problem*, submitted, available at https://arxiv.org/abs/1507.06289

[2] A.V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pac. J. Math. **10** (1960), 419–437.

[3] X. Cabré, J. Tan, , *Positive solutions of nonlinear problems involving the square root of the* , Adv. Math. **224** (2010), no. 5, 2052–2093.

[4] L. Caffarelli, , *Surfaces minimizing nonlocal energies*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **20** (2009), 281–299.

[5] L. Caffarelli, L. Silvestre, , *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7–9 1245–1260.

[6] A. Einstein, *´Uber die von der molekularkinetischen Theorie der W´arme gefordete Bewegung von in ruhenden Flu´ussigkeiten suspendierten Teilchen*, Ann. Phys. **322** (1905), 549–560.

[7] A. Fiscella, R. Servadei and E. Valdinoci, *Density properties for fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 235–253.

[8] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet forms and symmetric Markov processes. Walter De Gruyter, Berlin, 1994.

[9] P. Magrone, D. Mugnai, R. Servadei , *Multiplicity of solutions for semilinear variational inequalities via linking and ∇-theorems*, J. Differential Equations, **228** (2006), 191–225.

[10] A. Marino and C. Saccon, *Asymptotically critical points and multiple elastic bounce trajectories*, Topol. Methods Nonlinear Anal. **30** (2007), no. 2, 351–395.

[11] A. Marino and C. Saccon, *Some variational theorems of mixed type and elliptic problems with jumping nonlinearities*, Ann. Scuola Norm. Sup. Pisa Cl. Sc. (4) **XXV** (1997), Fasc. 3-4, 661–665.

[12] C. Mercier, *The magnetohydrodynamic approach to the problem of plasma confinement in closed magnetic configurations*, Euratom, Luxembourg (1974).

[13] A.M. Micheletti, A. Pistoia, C. Saccon, *Three solutions of a fourth order elliptic problem via variational theorems of mixed type*, Appl. Anal. **75** (2000), 43–59.

[14] G. Molica Bisci, D. Mugnai and R. Servadei, *On multiple solutions for nonlocal fractional problems via ∇-theorems*, Differential Integral Equations **30** (2017), 641–666.

[15] D. Mugnai, , *Four nontrivial solutions for subcritical exponential equations*, Calc. Var. Partial Differential Equations, **32** (2008), 481–497.

[16] D. Mugnai, , *Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem*, NoDEA Nonlinear Differential Equations Appl. **19** (2004), 379–391.

[17] D. Mugnai, *On a "reversed" variational inequality*, Top. Meth. Nonlin. Anal. **7** (2001), 321–358.

[18] D. Mugnai, D. Pagliardini, *Existence and multiplicity results for the fractional Laplacian in bounded domains*, Adv. Calc. Var. **10** (2017), 111–124.

[19] Z.-Q. Ou, C. Li, *Existence of three nontrivial solutions for a class of superlinear elliptic equations*, J. Math. Anal. Appl., **390** (2012), 418–426.

[20] P.H. Rabinowitz, *Some critical point theorems and applications to semilinear elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **4, 5** (1978), 215–223.

[21] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for nonlocal critical equations in low dimension*, Commun. Pure Appl. Anal. **12** (2013), 2445–2464.

[22] R. Servadei and E. Valdinoci, *Mountain Pass solutions for nonlocal elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.

[23] R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), 831–855.

[24] R. Servadei and E. Valdinoci, *Variational methods for nonlocal operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), 2105–2137.

[25] M. von Smoluchowski, *Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen*, Ann. Phys. **326** (1906), 756–780.

[26] Y. Sire, E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), 1842–1864.

[27] R. Song, Z. Vondraček, , *Potential theory of subordinate killed Brownian motion in a domain*, Probab. Theory Relat. Fields **125** (2003), no. 4, 578–592.

[28] R. Temam, *A non-linear eigenvalue problem: the shape at equilibrium of a confined plasma*, Arch. Rational Mech. Anal. **60** (1975), 51-73.

[29] R. Temam, *Remarks on a free boundary value problem arising in plasma physics*, Comm. Partial Differential Equations **2** (1977), 563–585.

[30] J.L. Vázquez, *Nonlinear Diffusion with Fractional Laplacian Operators*, Abel Symposia, **7** (2012), 271–298.

[31] F. Wang, *Multiple solutions for some nonlinear Schrdinger equations with indefinite linear part*, J. Math. Anal. Appl., **331** (2007), 1001–1022.

[32] W. Wang, A. Zang, P. Zhao, *Multiplicity of solutions for a class of fourth elliptic equations*, Nonlinear Anal., **70** (2009), 4377–4385.